The study of infinite series involving Smarandache function is one of the most interesting aspects of analysis.

In this brief article we give only a bare introduction to it.

First we prove that the series \( \sum_{k=2}^{\infty} \frac{S(k)}{(kH)!} \) converges and has the sum \( e^{e-\frac{5}{2}}, \frac{1}{2} \):

\[ S(m) \text{ is the Smarandache function: } S(m) = \min\{k \in \mathbb{N}; m|k!\} \, . \]

Let us denote \( 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} \) by \( E_n \). We show that

\[ E_{n+1} - \frac{5}{2} < \sum_{k=2}^{n} \frac{S(k)}{(k+1)!} < \frac{1}{2} \]

as follows:

\[ \sum_{k=2}^{n} \frac{k}{(k+1)!} = \sum_{k=2}^{n} \left( \frac{1}{k!} - \frac{1}{(k+1)!} \right) = \sum_{k=2}^{n} \frac{1}{k!} - \sum_{k=2}^{n} \frac{1}{(k+1)!} = \frac{1}{2!} - \frac{1}{(n+1)!} \]

\( S(k) < k \) implies that

\[ \sum_{k=2}^{n} \frac{S(k)}{(k+1)!} \leq \sum_{k=2}^{n} \frac{k}{(k+1)!} = \frac{1}{2} - \frac{1}{(k+1)!} < \frac{1}{2} \, . \]
On the other hand, \( k > 2 \) implies that \( S(k) > 1 \) and consequently:

\[
\sum_{k=2}^{n} \frac{S(k)}{(k+1)!} > \sum_{k=2}^{n} \frac{1}{(k+1)!} = \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{n+1}! = E_{n+1} - \frac{5}{2}.
\]

It follows that \( E_{n+1} - \frac{5}{2} < \sum_{k=2}^{n} \frac{S(k)}{(k+1)!} < \frac{1}{2} \) and therefore:

\[
\sum_{k=2}^{n} \frac{S(k)}{(k+1)!}
\]

is a convergent series with sum \( \sigma \in \left( e^{-\frac{5}{2}}, \frac{1}{2} \right) \).

**Remark:** Some of inequalities \( S(k) \leq k \) are strictly and \( k > S(k) + 1 \), \( S(k) > 2 \). Hence \( \sigma \in \left( e^{-\frac{5}{2}}, \frac{1}{2} \right) \).

We can also check that \( \sum_{k=r}^{n} \frac{S(k)}{(k-r)!} \), \( r \in \mathbb{N}^* \) and \( \sum_{k=r}^{n} \frac{S(k)}{(k+r)!} \), \( r \in \mathbb{N} \),

are both convergent as follows:

\[
\sum_{k=r}^{n} \frac{S(k)}{(k-r)!} \leq \sum_{k=r}^{n} \frac{k}{(k-r)!} = \frac{r}{0!} + \frac{r+1}{1!} + \frac{r+2}{2!} + \ldots + \frac{r+(n-r)}{(n-r)!} = \frac{r}{0!} + \frac{1}{1!} + \ldots + \frac{1}{(n-r)!} = rE_{n-r} - E_{n-r-1}
\]

We get \( \sum_{k=r}^{n} \frac{S(k)}{(k-r)!} < rE_{n-r} - E_{n-r-1} \) which that \( \sum_{k=r}^{n} \frac{S(k)}{(k+1)!} \)

converges.

Also we have \( \sum_{k=n}^{\infty} \frac{S(k)}{(k+r)!} < \infty \), \( r \in \mathbb{N} \).

Let us define the set \( \mathcal{M}_2 = \{ m \in \mathbb{N} : m = \frac{n!}{2}, n \in \mathbb{N}, n \geq 3 \} \).

If \( m \in \mathcal{M}_2 \) it is obvious that \( S(m) = n, \ m = \frac{n!}{2}, m \in \mathcal{M}_2 \rightarrow \frac{m}{S(m)!} = \frac{n!}{2} \).

So, \( \sum_{m \in \mathcal{M}_2} \frac{m}{S(m)!} = \infty \) and therefore \( \sum_{k \in \mathbb{N}} \frac{k}{S(k)!} = \infty \).

A problem: test the convergence behaviour of the series \( \sum_{k \in \mathbb{N}} \frac{1}{S(k)!} \).
REFERENCES


Current Address: Dept. of Math. University of Craiova,
Craiova (1100) - Romania