ON THE 49-TH SMARANDACHE’S PROBLEM*

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ABSTRACT. For any prime number \( p \) and positive integer \( n \), let \( S_p(n) \) be the smallest integer such that \( S_p(n)! \) is divisible by \( p^n \). In this paper, we study the mean value of the Dirichlet series with coefficients \( S_p(n) \). We also show that \( S_p(n) \) closely relates to Riemann Zeta function, and give a few asymptotic formulae involving \( S_p(n) \) and other arithmetic functions.

1. INTRODUCTION AND RESULTS

For any prime number \( p \) and positive integer \( n \), let \( S_p(n) \) be the smallest integer such that \( S_p(n)! \) is divisible by \( p^n \). For example, \( S_3(1) = 3, S_3(2) = 6, S_3(3) = 9, S_3(4) = 9, S_3(5) = 12, S_3(6) = 15, S_3(7) = 18, \ldots \) It is obvious that \( p \mid S_p(n) \) and \( S_p(n) \leq np \). Professor F. Smarandache [1] asks us to study the sequence. About this problem, we know very little. The problem is interesting because it can help us to calculate the Smarandache function.

It seems that \( S_p(n) \) closely relates to Riemann Zeta function. In fact, for real \( s > 1 \), we consider the Dirichlet series with coefficients \( S_p(n) \). The series \( \sum S_p(n)n^{-s} \) converges absolutely as \( s > 2 \) since \( S_p(n) \leq np \). In this paper, we study the mean value of the Dirichlet series with coefficients \( S_p(n) \), and give a few asymptotic formulae involving \( S_p(n) \) and other arithmetic functions.

Theorem 1. For any given \( s \), we have

\[
\sum_{n=1}^{\infty} \frac{S_p(n)}{n^s} = (p - 1)\zeta(s - 1) + R_1(s, p), \quad s > 2;
\]

\[
\sum_{n=1}^{\infty} \frac{\phi(n)S_p(n)}{n^s} = \frac{(p - 1)\zeta(s - 2)}{\zeta(s - 1)} + R_2(s, p), \quad s > 3,
\]

where

\[
R_1(s, p) \leq \frac{p - 1}{\log 2} \sum_{n=1}^{\infty} \frac{\log n + \log p}{n^s}, \quad R_2(s, p) \leq \frac{p - 1}{\log 2} \sum_{n=1}^{\infty} \frac{\phi(n)(\log n + \log p)}{n^s}.
\]

From our theorem we know that \( S_p(n) \) closely relates to Riemann Zeta function. Using our formulae we can calculate the mean value of \( S_p(n) \).

Key words and phrases. Primitive numbers; Dirichlet series; Mean value; Asymptotic formula.

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2. SOME LEMMAS

To complete the proof of the theorem, we need the following lemma.

**Lemma 1.** For any prime number $p$ and positive integer $n$, let $S_p(n)$ be the smallest integer such that $S_p(n)!$ is divisible by $p^n$. Then we have

$$n(p - 1) \leq S_p(n) \leq \left( n + \frac{\log(np)}{\log 2} \right) (p - 1).$$

**Proof.** By Theorem 3.14 of [2] we have

$$S_p(n)! = \prod_{p_1 \leq S_p(n)} p_1^{\alpha(p_1)}, \quad \alpha(p_1) = \sum_{m=1}^{\infty} \left\lfloor \frac{S_p(n)}{p_1^m} \right\rfloor,$$

where $\prod_{p_1 \leq x}$ denotes the product over prime numbers not exceeding $x$. Note that $p^n | S_p(n)$, we get

$$n \leq \alpha(p) = \sum_{m=1}^{\infty} \left\lfloor \frac{S_p(n)}{p^m} \right\rfloor \leq \sum_{m=1}^{\infty} \frac{S_p(n)}{p^m} = \frac{S_p(n)}{p - 1}.$$

On the other hand, $p^n \mid (S_p(n) - 1)!$ since $p \mid S_p(n)$. Therefore

$$n - 1 \geq \sum_{m=1}^{\infty} \left\lfloor \frac{S_p(n) - 1}{p^m} \right\rfloor \geq \sum_{m=1}^{\infty} \frac{S_p(n) - 1}{p^m} - \sum_{m=1}^{\infty} \frac{1}{p^m} \leq \frac{S_p(n) - 1}{p - 1} - \frac{\log(np)}{\log 2}.$$

So we have

$$S_p(n) \leq \left( n - 1 + \frac{\log(np)}{\log 2} \right) (p - 1) + 1 \leq \left( n + \frac{\log(np)}{\log 2} \right) (p - 1).$$

This proves Lemma 1.

3. PROOF OF THE THEOREM

In this section, we complete the proof of Theorem 1. From Lemma 1 we have

$$S_p(n) = n(p - 1) + O \left( (p - 1)(\log n + \log p) \right).$$

From Theorem 3.2 of [2] we immediately get

$$\sum_{n=1}^{\infty} \frac{S_p(n)}{n^s} = (p - 1)\zeta(s - 1) + R_1(s, p), \quad s > 2,$$

where

$$R_1(s, p) \leq \frac{p - 1}{\log 2} \sum_{n=1}^{\infty} \frac{\log n + \log p}{n^s}.$$

Similarly we can deduce other formula.

This completes the proof of Theorem 1.

REFERENCES