On the 57-th Smarandache's problem *

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Abstract

For any positive integer \( n \), let \( r \) be the positive integer such that:
the set \( \{1, 2, \ldots, r\} \) can be partitioned into \( n \) classes such that no
class contains integers \( x, y, z \) with \( xy = z \). In this paper, we use the
elementary methods to give a sharp lower bound estimate for \( r \).

§1. Introduction

For any positive integer \( n \), let \( r \) be a positive integer such that:
the set \( \{1, 2, \ldots, r\} \) can be partitioned into \( n \) classes such that no
class contains integers \( x, y, z \) with \( xy = z \). In [1], Schur asks us to
find the maximum \( r \). About this problem, it appears that no one had
studied it yet, at least, we have not seen such a paper before. The
problem is interesting because it can help us to study some important
partition problem. In this paper, we use the elementary methods to
study Schur’s problem and give a sharp lower bound estimate for \( r \).
That is, we shall prove the following:

**Theorem** For sufficiently large integer \( n \), let \( r \) be a positive integer such that:
the set \( \{1, 2, \ldots, r\} \) can be partitioned into \( n \) classes such that no
class contains integers \( x, y, z \) with \( xy = z \). For any
number \( \varepsilon > 0 \), We have

\[
r \geq n^{2(1-\varepsilon)(n-1)}.
\]

Whether the upper bound of \( r \) is \( n^{2(n-1)} \), or there exists another
sharper lower bound estimate for \( r \), is an interesting problem.

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§2. Proof of the Theorem

In this section, we complete the proof of the Theorem.

Let \( r = \left\lfloor n^{2(1-\varepsilon)(n-1)} \right\rfloor \) and partition the set \( \{1, 2, \ldots, \left\lfloor n^{2(1-\varepsilon)(n-1)} \right\rfloor \} \) into \( n \) classes as follows:

Class 1: \( 1, \left\lfloor n^{(1-\varepsilon)(n-1)} \right\rfloor, \left\lfloor n^{(1-\varepsilon)(n-1)} + 1 \right\rfloor, \ldots, \left\lfloor n^{2(1-\varepsilon)(n-1)} \right\rfloor \).

Class 2: \( n + 1, n + 2, \ldots, \left\lfloor \frac{n^{(1-\varepsilon)(n-1)} - 1}{n(n-1)\cdots3} \right\rfloor \).

Class 3: \( \left\lfloor \frac{n^{(1-\varepsilon)(n-1)} - 1}{n(n-1)\cdots4} \right\rfloor + 1, \left\lfloor \frac{n^{(1-\varepsilon)(n-1)} - 1}{n(n-1)\cdots3} \right\rfloor + 2, \ldots, \left\lfloor \frac{n^{(1-\varepsilon)(n-1)} - 1}{n(n-1)\cdots4} \right\rfloor \).

\vdots

Class \( k \): \( \left\lfloor \frac{n^{(1-\varepsilon)(n-1)} - 1}{n(n-1)\cdots(k+1)} \right\rfloor + 1, \left\lfloor \frac{n^{(1-\varepsilon)(n-1)} - 1}{n(n-1)\cdots k} \right\rfloor + 2, \ldots, \left\lfloor \frac{n^{(1-\varepsilon)(n-1)} - 1}{n(n-1)\cdots(k+1)} \right\rfloor \).

\vdots

Class \( n \): \( n, \left\lfloor \frac{n^{(1-\varepsilon)(n-1)} - 1}{n} \right\rfloor + 1, \left\lfloor \frac{n^{(1-\varepsilon)(n-1)} - 1}{n} \right\rfloor + 2, \ldots, \left\lfloor n^{(1-\varepsilon)(n-1)} - 1 \right\rfloor \).

where \( \lfloor y \rfloor \) denotes the integer part of \( y \).

It is obvious that Class \( k \) contains no integers \( x, y, z \) with \( xy = z \) for \( k = 1, 3, 4, \ldots, n \). In fact for any integers \( x, y, z \in \text{Class } k, k = 3, 4, \ldots, n \), we have

\[ xy \geq k \times \left( \left\lfloor \frac{n^{(1-\varepsilon)(n-1)} - 1}{n(n-1)\cdots k} \right\rfloor + 1 \right) > \left\lfloor \frac{n^{(1-\varepsilon)(n-1)} - 1}{n(n-1)\cdots(k+1)} \right\rfloor \geq z. \]

On the other hand, \( \left\lfloor \frac{n^{(1-\varepsilon)(n-1)} - 1}{n(n-1)\cdots3} \right\rfloor \) tends to zero when \( n \to \infty \), so for sufficiently large integer \( n \), Class 2 has only one integer 2.

This completes the proof of the Theorem.

References


