ON THE EQUATION $S(mn)=m^kS(n)$

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Abstract. In this paper we prove that the equation $S(mn)=m^kS(n)$ has only the positive integer solution $(m,n,k)=(2,2,1)$ with $m>1$ and $n>1$.

Key words Smarandache function, equation, positive integer solution.

For any positive integer $a$, let $S(a)$ be the Smarandache function. Muller [2, Problem 21] proposed a problem concerning the integer solutions $(m,n,k)$ of the equation (1) $S(mn)=m^kS(n), \ m>1, n>1$.

In this paper we determine all solutions of (1) as follows.

Theorem. The equation (1) has only the solution $(m,n,k)=(2,2,1)$.

Proof. By [1, Theorem], we have

(2) $S(mn) \leq S(m)+S(n)$.

Hence, if $(m,n,k)$ is a solution of (1), then from (2) we obtain

(3) $m^kS(n) \leq S(m)+S(n)$.

By (3), we get

(4) $m^k \leq \frac{S(m)}{S(n)} + 1$.

Since $S(m) \leq m$, we see from (4) that

(5) $m^k \leq \frac{m}{S(n)} + 1$.

If $n>2$, then $S(n) \geq 3$ and

(6) $m \leq mk \leq \frac{m}{3} + 1$,

by (5). However, we get from (6) that $m \leq 1/2$, a
contradiction. So we have $n=2$. Then, we get $S(n)=2$ and

$$m^k \leq \frac{m}{2} + 1 \tag{7}$$

by (5).

If $m > 2$, then $m/2 > 1$, and

$$m \leq m^k - \frac{m}{2} + \frac{m}{2} = m \tag{8}$$

by (7). This is a contradiction. Therefore, we get $m = 2$ and

$$2^k \leq 1 + 1 = 2, \tag{9}$$

by (7). Thus, we see from (9) that (1) has only the solution $(m, n, k) = (2, 2, 1)$. The theorem is proved.

References


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