On The Functional Equation $Z(n) + \varphi(n) = d(n)$

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Abstract: For any positive integer $n$, let $d(n)$, $\varphi(n)$ and $Z(n)$ denote the divisor function, the Euler function and the pseudo-Smarandache function of $n$ respectively. In this paper, we prove that the functional equation $Z(n) + \varphi(n) = d(n)$ has no solution $n$.

Key words: divisor function, Euler function, pseudo-Smarandache function.

Let $N$ be the set of all positive integers. For any positive integer $n$, let

(1) \[ d(n) = \sum_{d|n} 1, \]
(2) \[ \varphi(n) = \sum_{1 \leq m \leq n, \gcd(m,n)=1} 1, \]
(3) \[ Z(n) = \min \left\{ a \mid a \in N, n | \sum_{j=1}^{a} j \right\}. \]

Then $d(n)$, $\varphi(n)$ and $Z(n)$ are called the divisor function, the Euler function and the Pseudo-Smarandache function of $n$ respectively. In [1], Ashbacher posed the following unsolved question:

Question: How many solutions $n$ are there to the functional equation

(4) \[ Z(n) + \varphi(n) = d(n), \quad n \in N? \]

In this paper, we completely solve the above-mentioned question as follows:

Theorem: The equation $Z(n) + \varphi(n) = d(n), \quad n \in N$ has no solution. \[ \]

Proof: Let $n$ be a solution of (4). A computer search showed that (4) has no solution with $n \leq 10000$ (see [1]). So we have $n > 10000$. Let

(5) \[ n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \]

be the prime factorization of $n$. By [2, theorems 62 and 273], we see from (1), (2) and (5) that

(6) \[ d(n) = (r_1 + 1)(r_2 + 1) \cdots (r_k + 1) \]
(7) \[ \varphi(n) = n \prod_{i=1}^{k} \left( 1 - 1/p_i \right) \]

On the other hand, it is a well-known fact that
From (8) we get
\[ Z(n) \geq \sqrt{2n + \frac{1}{4} - \frac{1}{2}}. \]

Therefore, by (4), (5), (6), (7) and (9), we obtain
\[ f(n) + g(n) \geq 1. \]

where
\[
\begin{align*}
(11) & \quad f(n) = \prod_{i=1}^{k} \left(1 - \frac{1}{p_{i}}\right) \left(p_{i}^{n_{i}}/(r_{i} + 1)\right), \\
(12) & \quad g(n) = \sqrt{2} \prod_{i=1}^{k} \left(p_{i}^{n_{i}^{2}}/(r_{i} + 1)\right) - \frac{1}{2} \prod_{i=1}^{k} 1/(r_{i} + 1).
\end{align*}
\]

Clearly, we see from (12) that \( g(n) > 0 \) for any positive integer \( n \) with \( n > 1 \). Hence, we get from (10) that
\[ f(n) < 1. \]

If \( k = 1 \), then \( n = p_{1}^{n_{1}} \) and \( Z(n) \geq p_{1}^{n_{1}} - 1 \) by (3). Hence, by (1) and (2), \( n \) is not a solution of (4). This implies that \( k \geq 2 \).

If \( k \geq 3 \), then \( p_{k} \geq 5 \) and \( f(n) \geq 1 \), by (11). This contradicts with (13). So we have \( k = 2 \). Then (11) can be written as
\[ f(n) = (1 - 1/p_{1}) (1 - 1/p_{2}) \left((p_{1}^{n_{1}}p_{2}^{n_{2}})/(r_{1}+1)(r_{2}+1)\right). \]

If \( p_{2} > 3 \), then from (14) we get \( f(n) \geq 1 \), a contradiction. Hence, we deduce that \( p_{1} = 2 \) and \( p_{2} = 3 \). Then, by (13) and (14), we obtain
\[ f(n) = (2^{n_{1}}3^{n_{2}})/(3(r_{1}+1)(r_{2}+1)) < 1. \]

From (15), we can calculate that \( (r_{1},r_{2}) = (1,1) \) or \( (2,1) \). This implies that \( n \leq 12 \), a contradiction. Thus, (4) has no solution \( n \). The theorem is proved.

References


(2) G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, 1937.