ON THE PERFECT SQUARES IN SMARANDACHE CONCATENATED SQUARE SEQUENCE

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Abstract

Let $n$ be positive integer, and let $s(n)$ denote the $n$-th Smarandache concatenated square number. In this paper we prove that if $n = 2, 3, 4, 7, 8, 9, 11, 12, 14, 16, 17, 18, 20, 21, 22,$ or $25$ (mod 27), then $s(n)$ is not a square.

In [1], Marimutha defined the Smarandache concatenated square sequence $\{s(n)\}_{n=1}^{\infty}$ as follows:

\begin{align*}
(1) \quad & s(1) = 1, \quad s(2) = 14, \quad s(3) = 149, \quad s(4) = 14916, \\
& \quad s(5) = 1491625, \ldots.
\end{align*}

Then we called $s(n)$ the $n$-th Smarandache concatenated square number. Marimutha [1] conjectured that $s(n)$ is never a perfect square. In this paper we prove the following result:

Theorem.

If $n = 2, 3, 4, 7, 8, 9, 11, 12, 14, 16, 17, 18, 20, 21, 22,$ or $25$ (mod 27), then $s(n)$ is not a perfect square.

The above result implies that the density of perfect squares in Smarandache concatenated square sequence is at most $11/27$.

Proof of Theorem. We now assume that $s(n)$ is a perfect square. Then we have

\begin{align*}
(2) \quad & s(n) = x^2,
\end{align*}

were $x$ is a positive integer. Notice that $10^k \equiv 1 \pmod{9}$ for any positive integer $k$. We get from (1) and (2) that

\begin{align*}
(3) \quad & s(n) = 1^2 - 2^2 + \ldots - n^2 = 1/6 \cdot n(n-1)(2n+1) \equiv x^2 \pmod{9}.
\end{align*}

It implies that
\[(4) \quad n(n-1)(2n-1) \equiv 6x^2 \pmod{27}.
\]

If \( n \equiv 2 \pmod{27} \), then from (4) we get \( 2*3*5 \equiv 6x^2 \pmod{27} \). It follows that

\[(5) \quad x^2 \equiv 5 \pmod{9}.
\]

Since 5 is not a square residue mod 3, (5) is impossible. Therefore, if \( n \equiv 2 \pmod{27} \), then \( s(n) \) is not a square.

By using some similarly elementary number theory methods, we can check that the congruence (4) does not hold for the remaining cases. The theorem is proved.

Reference: