On the Pseudo-Smarandache Function

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Kashihara[2] defined the Pseudo-Smarandache function $Z$ by

$$Z(n) = \min \left\{ m \geq 1 : n \mid \frac{m(m+1)}{2} \right\}$$

Properties of this function have been studied in [1], [2] etc.

1. By answering a question by C. Ashbacher, Maohua Le proved that $S(Z(n)) - Z(S(n))$ changes signs infinitely often. Put

$$\Delta_{S,Z}(n) = | S(Z(n)) - Z(S(n)) |$$

We will prove first that

$$\liminf_{n \to \infty} \Delta_{S,Z}(n) \leq 1 \quad (1)$$

and

$$\limsup_{n \to \infty} \Delta_{S,Z}(n) = +\infty \quad (2)$$

Indeed, let $n = \frac{p(p+1)}{2}$, where $p$ is an odd prime. Then it is not difficult to see that $S(n) = p$ and $Z(n) = p$. Therefore,

$$| S(Z(n)) - Z(S(n)) | = | S(p) - S(p) | = | p - (p-1) | = 1$$

implying (1). We note that if the equation $S(Z(n)) = Z(S(n))$ has infinitely many solutions, then clearly the $\liminf$ in (1) is 0, otherwise is 1, since $S(Z(n)) - Z(S(n))$ being an integer.

Now let $n = p$ be an odd prime. Then, since $Z(p) = p-1$, $S(p) = p$ and $S(p-1) \leq \frac{p-1}{2}$
(see [4]) we get

\[ \Delta_{ef}(p) = \left| S(p-1) - (p-1) \right| = p-1 - S(p-1) \geq \frac{p-1}{2} \rightarrow \infty \text{ as } p \rightarrow \infty \]

proving (2). Functions of type \( \Delta_{ef} \) have been studied recently by the author [5] (see also [3]).

2. Since \( n \bigg| \frac{(2n-1)2n}{2} \), clearly \( Z(n) \leq 2n-1 \) for all \( n \).

This inequality is best possible for even \( n \), since \( Z(2^k) = 2^{k+1} - 1 \). We note that for odd \( n \), we have \( Z(n) \leq n - 1 \), and this is best possible for odd \( n \), since \( Z(p) = p-1 \) for prime \( p \). By

\[ \frac{Z(n)}{n} \leq 2 - \frac{1}{n} \text{ and } \frac{Z(2^k)}{2^k} = 2 - \frac{1}{2^k} \]

we get

\[ \limsup_{n \rightarrow \infty} \frac{Z(n)}{n} = 2. \quad (3) \]

Since \( Z\left(\frac{p(p+1)}{2}\right) = p \), and \( \frac{p}{p(p+1)/2} \rightarrow 0 \) \((p \rightarrow \infty)\), it follows

\[ \liminf_{n \rightarrow \infty} \frac{Z(n)}{n} = 0 \quad (4) \]

For \( Z(Z(n)) \), the following can be proved. By

\[ Z(Z\left(\frac{p(p+1)}{2}\right)) = p-1 \), clearly

\[ \liminf_{n \rightarrow \infty} \frac{Z(Z(n))}{n} = 0 \quad (5) \]

On the other hand, by \( Z(Z(n)) \leq 2Z(n) - 1 \) and (3), we have

\[ \limsup_{n \rightarrow \infty} \frac{Z(Z(n))}{n} \leq 4 \quad (6) \]

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3. We now prove

\[
\liminf_{n \to \infty} \left| Z(2n) - Z(n) \right| = 0 \quad (7)
\]

and

\[
\limsup_{n \to \infty} \left| Z(2n) - Z(n) \right| = +\infty \quad (8)
\]

Indeed, in [1] it was proved that \( Z(2p) = p-1 \) for a prime \( p \equiv 1 \pmod{4} \). Since \( Z(p) = p-1 \), this proves relation (7).

On the other hand, let \( n = 2^k \). Since \( Z(2^k) = 2^{k+1} - 1 \) and \( Z(2^{k+1}) = 2^{k+2} - 1 \), clearly \( Z(2^{k+1}) - Z(2^k) = 2^{k+1} \to \infty \) as \( k \to \infty \).

References