In this paper are presented the parametric solutions for the homogeneous diophantine equations:

\[ x^2 + by^2 + cz^2 = w^2 \]  

where \( b, c \) are rational integers.

I. Present theory.

Case 1: \( b = c = 1 \)

Curmichael [2] showed that the solutions are expressions with the form:

\[
\begin{align*}
  w &= p^2 + q^2 + u^2 + v^2; \\
  y &= 2pq + 2uv; \\
  x &= p^2 - q^2 + u^2 - v^2; \\
  z &= 2pv - 2qu;
\end{align*}
\]

where \( p, q, u, v \) are rational integers.

Mordell [3] showed that only these are the equations solution's by applying the arithmetic theory of the Gaussian integers.

Case 2: \( b = 1; \ c = -1 \). Mordell [3] showed that the solutions are, and only these, the expressions:

\[
\begin{align*}
  2x &= ad - bc; \\
  2y &= ac + bd; \\
  2z &= ac - bd; \\
  2w &= ad + bc;
\end{align*}
\]

\( a, b, c, d \) are integer parameters.

Case 3: \( b, c \) are rational integers.

Mordell [3] took the particular solutions with three parameters again, had been proposed by Euler:

\[
\begin{align*}
  w &= p^2 + bq^2 + cu^2; \\
  y &= 2pq; \\
  x &= p^2 - bq^2 - cu^2; \\
  z &= 2pu;
\end{align*}
\]

II. Results.

In [4] is proposed a new method to solve the quaternary equations using the notion of "quadratic combination". If we noted \( G_{ij} \), the complete system of equation's solutions: \( x^2 + y^2 = z^2 \), and also \( G_{ij} \) for the equation: \( x^2 + y^2 + z^2 = w^2 \), we shall can to enunciate:

Definition 1: Quadratic combination is a numerical function \( \Box \) which associates each two solutions from \( G_{ij} \), four solutions from \( G_{ij} \). Symbolically we have:

\[ \Box : G_{ij} \times G_{ij} \rightarrow G_{ij} \]
Observation.

From the quadratic combination of the equation’s solutions with the form: $x^2 + by^2 = z^2$, we shall obtain the solutions for the equations $x^2 + by^2 + cz^2 = w^2$ [4]

1. Case $b = c = 1$

From the quadratic combination, we find again the solution (2). We can present another demonstration for Mordell’s sentence. From [4] we have:

**Theorema 1.**

For the equation $E_j^2$, the solutions are expressions (2) and only these. The first part of the demonstration results by verification. For the second part of it, we can use the property demonstrated in [4].

**Lemma 2.** The multitude of the equation’s solutions $E_j^2$ is a graph $F_j^2$ as terminal top the ordinary solution $(1, 0, 0, 1)$ and the arcs are given by the “t” functions:

$t = w ± x ± y ± z$

The solutions are matriceally developed:

$$S_{pi} = S_{i} \cdot B$$

with $B = \begin{pmatrix} 0 & -1 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & -1 & -1 & 2 \end{pmatrix}$

**Lemma 3.** Any solutions from the equations (2) are on the graph $F_j^2$ and, reciprocally, any solutions from the $F_j^2$ can be written with form (2).

It was defined the term: $t_i = x + y + z - w$; $t_{i+1} < t_i$, where variables are natural numbers [4].

We are verifying that form every solution of natural numbers can derive a solution with $w_i < w$. The parameter’s correspondence ($p > q$ and $u > v$) will be:

$$p_i = p - q - v; \quad u_i = u + q - v;$$
$$q_i = q; \quad v_i = v$$

It is obtained a number of decreasing values $w_i$, having as limit the ordinary solutions $(1, 0, 0, 1)$. Reciprocally, for every solution from the graph $F_j^2$ is obtained a number of parameterly solutions with $w_i$ breeder, in case $t_{i+1} > t_i$.

2. Case $b = 1$, $c = -1$. From quadratic combination results equations:

$$w = p^2 + q^2 - u^2 - v^2$$
$$x = p^2 - q^2 + u^2 + v^2$$
$$y = 2pq + 2uv$$
$$z = 2pv + 2qu$$

It can be showed that the Mordell’s solutions (3) are equivalent with solutions (6); the parameter’s equivalence is given by:

$$a = p + v; \quad b = p - v$$
c = q - u ; d = q + u

3. Case b, c are rationale integers. For simplicity, we shall treat in two subcases:

3a) b, c prime numbers. The quadratic combination will require the solutions:

\[ w = p^2 + bq^2 + cu^2 + bcv^2 \]
\[ x = p^2 - bq^2 - cu^2 + bcv^2 \]
\[ y = 2pq + 2cuv \]
\[ z = 2pu - 2bqv \] (7)

3b) b and c are compound numbers. For any decomposition: \( b = i \cdot j \) and \( c = l \cdot h \), where \( i, j, l, h \) are rationale integers, we have the general solutions with four parameters of the equation (1):

\[ w = ihp^2 + jhq^2 + jlu^2 + ilv^2 \]
\[ x = ihp^2 - jhq^2 + jlu^2 - ilv^2 \]
\[ y = 2hpq + 2luv \]
\[ z = 2ipv - 2jqu \] (8)

III. Applications We shall take again from [4] only the application of the numerical representations of exponent 2. It is well known the Fermont - Lagrange Theory.

**Theorema 2**
For any natural number it is at least a representation by sum of four whole number’s square rest:

\[ z = u^2 + v^2 + w^2 + t^2 \] (9)

Later on another Theory was demonstrated:

**Theorema 3**
For any natural number \( z \neq 2^{2k}(8l + 7) \) it is least a representation of three whole a numbers:

\[ z = u^2 + v^2 + w^2 \] (9')

Our theory allows us to enunciate a much stranger theory:

**Theorema 4**
For any natural number \( z \) it is at least three whole numbers \( (u, v, w) \) or \( (a, b, c) \), in order to have:

\[ z = u^2 + v^2 + w^2 \] (α)
\[ z = a^2 + b^2 + 2c^2 \] (β)

For \( z = z_1 = 2^{2k}(8l + 7) \), we have only the representation (β), for \( z = z_2 = 2^{2k+1}(8l + 7) \), we have only the representation (α) and for \( z \neq z_1 \neq z_2 \), we have in the same time the representations (α) and (β).
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