ON THE SMARANDACHE DOUBLE FACTORIAL FUNCTION

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Abstract. In this paper we discuss various problems and conjectures concerned the Smarandache double factorial function.

Keywords: Smarandache double factorial function, inequality, infinite series, infinite product, diophantine equation

For any positive integer \( n \), the Smarandache double factorial function \( Sd(n) \) is defined as the least positive integer \( m \) such that \( m!! \) is divisible by \( n \), where

\[
m!! = \begin{cases} 2.4...m, & \text{if } 2 \mid m, \\ 1.3...m, & \text{if } 2 \nmid m. \end{cases}
\]

In this paper we shall discuss various problems and conjectures concerned \( Sd(n) \).

1. The value of \( Sd(n) \)

By the definition of \( Sd(n) \), we have \( Sd(1)=1 \) and \( Sd(n)>1 \) if \( n \) > 1. We now give three general results as follows.

**Theorem 1.1.** If \( 2 \mid n \) and

\[
n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}
\]

(1.1)

is the factorization of \( n \), where \( p_1, p_2, \ldots, p_k \) are distinct odd primes and \( a_1, a_2, \ldots, a_k \) are positive integers, then
(1.2) \( Sdf(n) = \max\{Sdf(p_1^n), Sdf(p_2^{a_2}), \ldots, Sdf(p_k^{a_k})\} \)

Proof. Let \( m_i = sdf(p_i^{a_i}) \) for \( i = 1, 2, \ldots, k \). Then we get \( 2 \nmid m_i \) (\( i = 1, 2, \ldots, k \)) and

\[(1.3) \quad p_i^{a_i} \mid m_i!! \text{, } i = 1, 2, \ldots, k.\]

Further let \( m = \max(m_1, m_2, \ldots, m_k) \). Then we have

\[(1.4) \quad m_i!! \mid m!! \text{, } i = 1, 2, \ldots, k.\]

Therefore, by (1.3) and (1.4), we get

\[(1.5) \quad p_i^{a_i} \mid m!! \text{, } i = 1, 2, \ldots, k.\]

Notice that \( p_1, p_2, \ldots, p_k \) are distinct odd primes. We have

\[(1.6) \quad \gcd(p_i^{a_i}, p_j^{a_j}) = 1, 1 \leq i < j \leq k.\]

Thus, by (1.1), (1.5) and (1.6), we obtain \( n \mid m!! \). It implies that

\[(1.7) \quad Sdf(n) \leq m.\]

On the other hand, by the definition of \( m \), if \( Sdf(n) < m \), then there exists a prime power \( p_j^{a_j} (1 \leq j \leq k) \) such that

\[(1.8) \quad p_j^{a_j} \mid Sdf(n)!!.\]

By (1.1) and (1.8), we get \( n \mid Sdf(n)!! \), a contradiction. Therefore, by (1.7), we obtain \( Sdf(n) = m \). It implies that (1.2) holds. The theorem is proved.

Theorem 1.2. If \( 2 \mid n \) and

\[(1.9) \quad n = 2^a n_1,\]

where \( a, n_1 \) are positive integers with \( 2 \nmid n_1 \), then
Proof. Let \( m_0 = Sdf(2^a) \) and \( m_1 = Sdf(n_1) \). Then we have

\[
2^a | m_0!!, n_1 | m_1!!.
\]

Since \( (2m_1)!! = 2 \cdot 4 \cdots (2m_1) = 2^{m_1} \cdot m_1!! = m_1!!(m_1 - 1)!! \), we get \( m_1!!(2m_1)!! \). It implies that

\[
(1.12) \quad n_1 | (2m_1)!!.
\]

Let \( m = \max(m_0, 2m_1) \). Then we have \( m_0!! | m!! \) and \( (2m_1)!! | m!! \). Since \( \gcd(2^a, n_1) = 1 \), we see from (1.9), (1.11) and (1.12) that \( n | m!! \). Thus, we obtain \( Sdf(n) \leq m \). It implies that (1.10) holds. The theorem is proved.

**Theorem 1.3.** Let \( a, b \) be two positive integers. Then we have

\[
Sdf(ab) \leq \begin{cases} 
Sdf(a) + Sdf(b), & \text{if } 2 | a \text{ and } 2 | b, \\
Sdf(a) + 2Sdf(b), & \text{if } 2 | a \text{ and } 2 | b, \\
2Sdf(a) + 2Sdf(b) - 1, & \text{if } 2 | a \text{ and } 2 | b.
\end{cases}
\]

Proof. By Theorem 4.13 of [4], if \( 2 | a \) and \( 2 | b \), then

\[
Sdf(a) = 2r, \quad Sdf(b) = 2s,
\]

where \( r, s \) are positive integers. We see from (1.14) that

\[
(1.15) \quad a | (2r)!!, b | (2s)!!.
\]

Notice that

\[
(1.16) \quad \frac{(2r + 2s)!!}{(2r)!!(2s)!!} = \frac{2^{r+s} \cdot (r+s)!}{(2^r \cdot r!)(2^s \cdot s!)} = \frac{(r+s)!}{r!s!} = \binom{r+s}{r},
\]

where \( \binom{r+s}{r} \) is a binomial coefficient. Since \( \binom{r+s}{r} \) is a positive integer, we see from (1.16) that
Thus, by (1.15) and (1.17), we get $ab(2r+2s)!!$. It implies that

(1.18) \[ Sdf(ab) \leq 2r+2s, \text{ if } 2|a \text{ and } 2|b. \]

If $2|a$ and $2 \nmid b$, then

(1.19) \[ Sdf(a)=2r, Sdf(b)=2s+1, \]

where $a$ is a positive integer and $s$ is a nonnegative integer. By (1.19), we get

(1.20) \[ a|2r!!, b|(2s+1)!!. \]

Notice that

(1.21) \[
\frac{(2r+4s+2)!!}{(2r)!!(2s+1)!!} = \frac{2^{r+2s+1} \cdot (r+2s+1)! \cdot 2^s \cdot s!}{2^r \cdot r! \cdot (2s+1)!} = 2^{3s+1} \cdot \frac{(r+2s+1)!}{s! \cdot (r+2s+1)!} = 2^{3s+1} \cdot \frac{(r+2s+1)!}{r!(2s+1)!}.
\]

We find from (1.21) that

(1.22) \[ (2r)!!(2s+1)!!(2r+4s+2)!!. \]

Thus, by (1.20) and (1.22), we obtain $ab|(2r+4s+2)!!$. It implies that

(1.23) \[ Sdf(ab) \leq 2r+4s+2, \text{ if } 2|a \text{ and } 2|b. \]

If $2 | a$ and $2 | b$, then

(1.24) \[ Sdf(a)=2r+1, Sdf(b)=2s+1, \]

where $r, s$ are nonnegative integers. By (1.24), we get

(1.25) \[ a|(2r+1)!!, b|(2s+1)!!. \]

Notice that
\[(1.26) \quad \frac{(4r + 4s + 3)!}{(2r + 1)!(2s + 1)!} = \frac{(4r + 4s + 3)!}{(2r + 1)!} \cdot \frac{(2r)!}{(2s)!!} \cdot \frac{(2s)!}{(2r + 1)!} \]
\[
= \frac{(4r + 4s + 3)!}{2^{2r+2s+1} \cdot (2r + 2s + 1)!} \cdot \frac{2^r \cdot r!}{(2r + 1)!} \cdot \frac{2^s \cdot s!}{(2s + 1)!} \\
= \frac{r! \cdot s!}{2^{2r+2s+1}} \left( \frac{4r + 4s + 3}{2r + 2s + 1, 2r + 1, 2s + 1} \right),
\]
where \( \left( \frac{4r + 4s + 3}{2r + 2s + 1, 2r + 1, 2s + 1} \right) \) is a polynomial coefficient. Since
\[
\left( \frac{4r + 4s + 3}{2r + 2s + 1, 2r + 1, 2s + 1} \right) \text{ is a positive integer and } (2r+1)!, (2s+1)!!
\]
are odd integers, we see from (1.26) that
\[(1.27) \quad (2r+1)!(2s+1)!!(4r+4s+3)!!.\]
Thus, by (1.25) and (1.27), we get \( ab | (4r+4s+3)!! \). It implies that
\[(1.28) \quad Sd_{ab}(ab) \leq 4r+4s+3, \text{ if } 2 \mid a \text{ and } 2 \mid b.\]
The combination of (1.18), (1.23) and (1.28) yields (1.13). The \( n \) theorem is proved.

**Theorem 1.4** Let \( p \) be a prime and let \( a \) be a positive integer. The we have
\[(1.29) \quad p \mid Sd_{ab}(p^a).\]

**Proof.** Let \( m = Sd_{ab}(p^a) \). By Theorem 4.13 of [4], if \( p=2 \), then \( m \) is even. Hence, (1.29) holds for \( p=2 \). If \( p > 2 \), then \( m \) is an odd integer with
\[(1.30) \quad p^a \mid m!! \]
We now suppose that $p \mid m$. Let $t$ be the greatest odd integer such that $t < m$ and $p \nmid t$. Then we have

$$(1.31) \quad m!! = t!!(t+2)\cdots(m-2)m,$$

where $t+2, \cdots, m-2, m$ are integers satisfying $p \nmid(t+2)\cdots(m-2)m$. Therefore, by (1.30) and (1.31), we get

$$(1.32) \quad p^n \nmid t!!.$$

By (1.32), we get $m = Sdf(p^n) \leq t < m$, a contradiction. Thus, we obtain $p \nmid m$. The theorem is proved.

**Theorem 1.5** Let $p$ be the least prime divisor of $n$. Then we have

$$(1.33) \quad Sdf(n) \geq p.$$

**Proof.** Let $m = Sdf(n)$. By Theorem 4.13 of [4], if $2 \mid n$, then $p=2$ and $m$ is an even integer. So we get (1.33).

If $2 \nmid n$, let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1, p_2, \cdots, p_k$ are distinct odd primes and $a_1, a_2, \cdots, a_k$ are positive integers. By Theorem 1.1, we get

$$(1.34) \quad m = \max(Sdf(p_1^{a_1}), Sdf(p_2^{a_2}), \ldots, Sdf(p_k^{a_k})).$$

Further, by Theorem 1.4, we have $p_i \mid Sdf(p_i^{a_i})$ for $i=1, 2, \cdots, k$. It implies that $Sdf(p_i^{a_i}) \geq p_i$ for $i=1, 2, \cdots, k$. Thus, by (1.34), we obtain

$$(1.35) \quad m \geq \min(p_1, p_2, \cdots, p_k) = p.$$ 

The theorem is proved.

**Theorem 1.6** For any positive integer $n$, we have
\[(1.36) \quad Sdf(n!) = \begin{cases} n, & \text{if } n = 1, 2, \\ 2n, & \text{if } n > 2. \end{cases} \]

**Proof.** Let \( m = Sdf(n!) \). Then (1.36) holds for \( n = 1, 2 \). If \( n > 2 \), then both \( n! \) and \( m \) are even. Since \((2n)!! = 2^n n!\), we get
\[(1.37) \quad m \leq 2n.\]

If \( m < 2n \), then \( m = 2n - 2r \), where \( r \) is a positive integer. Since \( m = Sdf(n!) \),
\[(1.38) \quad \frac{(2n - 2r)!}{n!} = \frac{2^{n-r} \cdot (n - r)!}{2^{n-r}} = \frac{2^{n-r}}{(n - r + 1) \ldots (n-1)n} \]

must be an integer. But, since either \( n-1 \) or \( n \) is an odd integer greater than 1, it is impossible by (1.38). Thus, by (1.37), we obtain \( m = 2n \).

The theorem is proved.

**Theorem 1.7** The equality
\[(1.39) \quad Sdf(n) = n\]
holds if and only if \( n \) satisfies one of the following conditions:

(i) \( n = 1, 9 \).

(ii) \( n = p \), where \( p \) is a prime.

(iii) \( n = 2p \), where \( p \) is a prime.

**Proof.** Let \( m = Sdf(n) \). If \( 2 \mid n \), let \( n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k} \) be the factorization of \( n \). By Theorem 1.1, we (1.34). Further, by Theorem 4.7 of [4], we have
\[(1.40) \quad Sdf(p_i^{a_i}) \leq p_i^{a_i}, i = 1, 2, \ldots, k.\]

Therefore, by (1.34) and (1.40), we obtain
\[(1.41) \quad m \leq \max\{p_1^{a_1}, p_2^{a_2}, \ldots, p_k^{a_k}\} \]

It implies that if \( k > 1 \), then \( m < n \). If \( k = 1 \) and (1.39) holds, then
By Theorem 4.1 of [4], (1.42) holds for \( a_i = 1 \). Since 2 \mid n, \( p_1 \) is an odd prime. By Theorem 1.3, if (1.42) holds, then we have

\[(1.43) p_i^{a_i} = m = Sdf\left(p_i^{a_i}\right) = Sdf\left(p_i, p_i, \ldots, p_i\right) \leq 2a_i \cdot Sdf(p_i) - 1 = 2a_i p_i - 1\]

Since \( p_i \geq 3 \), (1.43) is impossible for \( a_i > 2 \). If \( a_i = 2 \), then from (1.43) we get

\[(1.44) p_i^2 \leq 4p_i - 1,\]

whence we obtain \( p_i = 3 \). Thus, (1.39) holds for an odd integer \( n \) if and only if \( n = 19 \) or \( p \), where \( p \) is an odd prime.

If 2\mid n, then \( n \) can be rewritten as (1.9), where \( n_1 \) is an odd integer with \( n_1 \geq 1 \). By Theorem 1.2, if (1.39) holds, then we have

\[(1.45) n = 2^a n_1 \leq \max(Sdf(2^n), 2Sdf(n_1)).\]

We see from (1.45) that if (1.39) holds, then either \( n_1 = 1 \) or \( a = 1 \).

When \( n_1 = 1 \), we get from (1.39) that \( a = 1 \) or 2. When \( a = 1 \), we get,

\[(1.46) 2n_1 = Sdf(2n_1).\]

It is a well known fact that if \( n_1 \) is not an odd prime, then there exists a positive integer \( t \) such that \( t \mid n_1 \) and \( n_1 \mid t! \). Since \((2t)! = 2^t \cdot t!\), we get

\[(1.47) Sdf(2n_1) \leq 2t \leq 2n_1,\]

a contradiction. Therefore, \( n_1 \) must be an odd prime. In this case, if \( Sdf(2n_1) < 2n_1 \), then \( Sdf(2n_1) = 2n_1 - 2r \), where \( r \) is a positive integer. But, since

\[(1.48) \frac{(2n_1 - 2r)!}{2n_1} = \frac{2^n \cdot (n_1 - r)}{2n_1} = \frac{2^{n-1-r} \cdot (n_1 - r)}{n_1}\]

is not an integer, it is impossible. Thus, (1.39) holds for an even
integer if and only if $n=2p$, where $p$ is a prime. The theorem is proved.

2. The inequalities concerned $Sdf(n)$

Let $n$ be a positive integer. In [4], Russo posed the following problems and conjectures.

(2.1) \[
\frac{n}{Sdf(n)} \leq \frac{n}{8} + 2
\]

**Problem 2.1.** Is the inequality true for any $n$?

**Problem 2.2.** Is the inequality

\[
\frac{n}{Sdf(n)} \geq \frac{1}{n^{0.73}}
\]
true for any $n$?

**Problem 2.3.** Is the inequality

\[
\frac{1}{n \cdot Sdf(n)} < n^{-5/4}
\]
true for any $n$?

**Problem 2.4.** Is the inequality

\[
\frac{1}{n} + \frac{1}{Sdf(n)} < n^{-1/4}
\]
true for any $n$ with $n>2$?

**Conjecture 2.1.** For any positive number $\varepsilon$, there exist some $n$ such that

\[
\frac{Sdf(n)}{n} < \varepsilon
\]

In this respect, Russo [4] showed that if $n \leq 1000$, then the
inequalities (2.1), (2.2), (2.3) and (2.4) are true. We now completely solve the above-mentioned questions as follows.

**Theorem 2.1.** For any positive integer \( n \), the inequality (2.1) is true.

**Proof.** We may assume that \( n > 1000 \). Since \( m!! \leq 945 \) for \( m = 1, 2, \ldots, 9 \), if \( n > 1000 \), then \( Sdf(n) \geq 10 \). So we have

\[
\frac{n}{Sdf(n)} \leq \frac{n}{10} < \frac{n}{8} + 2.
\]

(2.6)

It implies that (2.1) holds. The theorem is proved.

The above theorem shows that the answer of Problem 2.1 is "yes".

In order to solve Problems 2.2, 2.3 and 2.4, we introduce the following result.

**Theorem 2.2.** If \( n = (2r)!! \), where \( r \) is a positive integer with \( r \geq 20 \), then

\[
Sdf(n) < n^{0.1}.
\]

(2.7)

**Proof.** We now suppose that

\[
Sdf(n) \geq n^{0.1}.
\]

(2.8)

Since \( n = (2r)!! \), we get \( Sdf(n) = 2r \). Substitute it into (2.8), we obtain that if \( r \geq 20 \), then

\[
2r \geq ((2r)!!)^{0.1} = 2^{0.1} \cdot (r!)^{0.1} \geq 2^2 (r!)^{0.1}.
\]

(2.9)

By the Strling theorem (see [1]), we have

\[
r! > \sqrt{2\pi r} \left( \frac{r}{e} \right)^r.
\]

(2.10)

Since \( r \geq 20 \), we get \( r/e > \sqrt{r} \). Hence, by (2.9) and (2.10), we obtain
a contradiction. Thus, we get (2.7). The theorem is proved.

By the above theorem, we obtain the following corollary immediately.

**Corollary 2.1.** If \( n=(2r)!! \), where \( r \) is a positive integer with \( r \geq 20 \), then the inequalities (2.2), (2.3) and (2.4) are false.

The above corollary means that the answers of Problems 2.2, 2.3 and 2.4 are "no".

**Theorem 2.3.** For any positive number \( \epsilon \), there exist some positive integers \( n \) satisfy (2.5).

**Proof.** Let \( n=(2r)!! \), where \( r \) is a positive integer with \( r \geq 20 \). By Theorem 2.2, we have

\[
(2.12) \quad \frac{Sdf(n)}{n} < \frac{n^{0.1}}{n} = \frac{1}{n^{0.9}}.
\]

By (2.12), we get

\[
(2.13) \quad \lim_{r \to \infty} \frac{Sdf(n)}{n} = 0.
\]

Thus, by (2.13), the theorem is proved.

By the above theorem, we see that Conjecture 2.1 is true.

3. The difference \(|Sdf(n+1)-Sdf(n)|\)

In [4], Russo posed the following problem.

**Problem 3.1.** Is the difference \(|Sdf(n+1)-Sdf(n)|\) bounded or unbounded?
We now solve this problem as follows.

**Theorem 3.1.** The difference $|Sdf(n+1)-Sdf(n)|$ is unbounded.

**Proof.** Let $m$ be a positive integer, and let $p$ be a prime. Further let ord $(p, m!)$ denote the order of $p$ in $m$. For any positive integer $a$, it is a well known fact that

$$\text{ord}(p, a!) = \sum_{k=1}^{s} \left\lfloor \frac{a}{p^k} \right\rfloor$$

(see Theorem 1.11.1 of [3]).

Let $r$ be a positive integer. Then we have

$$2^r! = 2 \cdot 4 \cdot 2^r = 2^r \cdot 2^r!$$

where

$$s = 2^r - 1.$$

By (3.1), (3.2) and (3.3), we get

$$\text{ord}(2, 2^r!) = 2^r + \text{ord}(2, 2^r!) = 2^r + (2^r + 2^r + \cdots + 2^r + 1) = 2^r - 1.$$

Let $n = 2^r$, where $r = 2^r$. Then, by (3.4), we get

$$Sdf(n) = 2^r + 2.$$

On the other hand, then $n+1 = 2^r + 1$ is a Fermat number. By the proof of Theorem 5.12.1 of [3], every prime divisor $q$ of $n+1$ is the form $q = 2^r + 1$, where $l$ is a positive integer. It implies that

$$q \geq 2^r + 1.$$  

Since $n+1$ is an odd integer, by Theorem 1.4, we get from (3.6) that

$$Sdf(n+1) \geq q \geq 2^r + 1.$$

We see from (3.8) that the difference $|Sdf(n+1)-Sdf(n)|$ is unbounded. Thus, the theorem is proved.
4. Some infinite series and products concerned $Sdf(n)$

In [4], Russo posed the following problems.

**Problem 4.1.** Evaluate the infinite series

\[
S = \sum_{n=1}^{\infty} \frac{(-1)^n}{Sdf(n)}.
\]

**Problem 4.2.** Evaluate the infinite product

\[
P = \prod_{n=1}^{\infty} \frac{1}{Sdf(n)}.
\]

We now solve the above-mentioned problems as follows.

**Theorem 4.1.** $S=\infty$.

**Proof.** For any nonnegative integer $m$, let

\[
g(m) = \frac{-1}{Sdf(2m+1)} + \sum_{i=1}^{\infty} \frac{1}{Sdf(2i(2m+1))}.
\]

By (4.1) and (4.3), we get

\[
S = \sum_{m=0}^{\infty} g(m).
\]

We see from (4.3) that

\[
g(0) = -1 + \frac{1}{Sdf(2)} + \frac{1}{Sdf(4)} + \frac{1}{Sdf(8)} + \ldots
\]

\[
= -1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \ldots > \frac{1}{2}.
\]

For positive integer $m$, let $t=Sdf(2m+1)$. Then $t$ is an odd integer with $t \geq 3$. Notice that $2m+1 | t!!$ and

\[
(2t)!! = 2^t \cdot t!!.
\]

We get from (4.6) that $2^j(2m+1)(2t)!!$ for $j=1, 2, \ldots, t$. It implies that
\[(4.7) \quad S_{df}(2^t(2m+1)) \leq 2t, j = 1, 2, \ldots, t.\]

Therefore, by (4.3) and (4.7), we obtain
\[(4.8) \quad g(m) > -\frac{1}{t} + \frac{1}{2t} + \frac{1}{2t} + \frac{1}{2t} = \frac{1}{2t}.\]

On the other hand, by Theorem 4.7 of [4], we have \(t \leq 2m+1\). By (4.8), we get
\[(4.9) \quad g(m) > -\frac{1}{2(2m+1)}.\]

Thus, by (4.4), (4.5) and (4.9), we obtain
\[(4.10) \quad S > \frac{1}{6} + \sum_{m=1}^{\infty} \frac{1}{2(2m+1)} = \infty.\]

The theorem is proved.

**Theorem 4.2.** \(P = 0.\)

**Proof.** Since \(S_{df}(n) > 1\) if \(n > 1\), by (4.2), we get \(p = 0\) immediately.

The theorem is proved.

5. The diophantine equations concerned \(S_{df}(n)\)

Let \(N\) be the set of all positive integers. In [4], Russo posed the following problems.

**Problem 5.1** Find all the solutions \(n\) of the equation
\[(5.1) \quad S_{df}(n)! = S_{df}(n!), n \in N.\]

**Problem 5.2** Is the equation
\[(5.2) \quad (S_{df}(n))^k = k \cdot S_{df}(nk), n, k \in N, n > 1, k > 1\]
have solutions \((n, k)\)?

**Problem 5.3** Is the equation
Theorem 5.1 The equation (5.1) has only the solutions \( n = 1, 2, 3 \).

Proof. Clearly, (5.1) has solutions \( n = 1, 2, 3 \). We suppose that (5.1) has a solution \( n \) with \( n > 3 \). By Theorem 1.6, if \( n > 2 \), then

(5.4) \( Sdf(n)! = 2n \).

Substitute (5.4) into (5.1), we get

(5.5) \( Sdf(n)! = 2n \).

Let \( m = Sdf(n) \). If \( n > 3 \) and \( 2 \mid n \), then \( n \geq 5 \), \( m \geq 5 \) and \( 4 \mid m! \).

However, since \( 2 \mid 2n \), (5.5) is impossible.

If \( n > 3 \) and \( 2 \mid n \), then \( m = 2t \), where \( t \) is a positive integer with \( t > 1 \).

From (5.5), we get

(5.6) \( (2t)! = 2n \).

Since \( m = Sdf(n) \), we have \( n \mid (2t)! \). It implies that

\[
\frac{(2t)!}{n} = \frac{2 \cdot (2t)!}{(2t)!} = \frac{2 \cdot (2t)!}{(2t)!!(2t - 1)!!} = \frac{2}{(2t - 1)!!}
\]

must be an integer. But, since \( t > 1 \), it is impossible. Thus, (5.1) has no solutions \( n \) with \( n > 3 \). The theorem is proved.

Theorem 5.2 The equation (5.2) has only the solutions \( (n, k) = (2, 4) \) and \( (3, 3) \).

Proof. Let \( (n, k) \) be a solution of (5.2). Further, let \( m = Sdf(n) \). By Theorem 1.3, we get
\(Sdf(nk) < 2 \cdot Sdf(n) + 2 \cdot Sdf(k) \geq 2(m+k)\).

Hence, by (5.2) and (5.7), we obtain
\[
(5.8) \quad m^k < 2k(m+k), \quad m > 1, \quad k > 1.
\]

If \(m=2\), then from (5.8) we get \(k \leq 6\). Notice that \(n=2\) if \(m=2\). We find from (5.2) that if \(m=2\) and \(k \leq 6\), then (5.2) has only the solution \((n, k)=(2, 4)\)

If \(m=3\), then from (5.8) we get \(k \leq 3\). Since \(n=3\) if \(m=3\). We see from (5.2) that if \(m=2\) and \(k \leq 3\), then (5.2) has only the solution \((n, k)=(3, 3)\).

If \(m=4\), then from (5.8) we get \(k \leq 2\). Notice that \(n=4\) or 8 if \(m=4\) and \(n=5\) or 15 if \(m=5\). Then (5.2) has no solution \((n, k)\). Thus, (5.2) has only the solutions \((n, k)=(2, 4)\) and (3.3). The theorem is proved.

**Theorem 5.3.** All the solutions \((m, n, k)\) of (5.3) are given in the following four classes:

(i) \(m=1\), \(n\) and \(k\) are positive integers.

(ii) \(n=1, \ k=1, \ m=1, \ 9, \ p \) or \(2p\), where \(p\) is a prime.

(iii) \(m=2, \ k=1, \ n\) is 2 or an odd integer with \(n \geq 1\).

(iv) \(m=3, \ k=1, \ n=3\).

**Proof.** If \(m=1\), then (5.3) holds for any positive integers \(n\) and \(k\). By Theorem 1.7, if \(n=1\), then from (5.3) we get (ii). Thus, (i) and (ii) are proved.

Let \((m, n, k)\) be a solution of (5.3) satisfying \(m > 1\) and \(n > 1\). By Theorem 1.3, if \(2|m\) and \(2|n\), then we have
Further, by Theorem 4.7 of [4], \( Sdf(m) \leq m \). Therefore, by (5.3) and (5.9), we obtain
\[
(5.10) \quad m \geq (m^k - 1) Sdf(n).
\]
When \( n=2 \), we get from (5.10) that \( m=2 \) and \( k=1 \).
When \( n>2 \), we get \( Sdf(n) \geq 4 \) and (5.10) is impossible.

If \( 2|m \) and \( 2 \mid n \), then
\[
(5.11) \quad Sdf(mn) \leq Sdf(m) + 2 \cdot Sdf(n).
\]
Notice that \( m \geq 2 \), \( n \) is an odd integer with \( n \geq 3 \), \( Sdf(n) \geq 3 \). We obtain from (5.3) and (5.11) that
\[
(5.12) \quad m \geq Sdf(m) \geq (m^k - 2) Sdf(n) \geq 3(m^k - 2) \geq 3(m - 2).
\]
From (5.12), we get \( m=2 \). Then, by (5.3), we obtain
\[
(5.13) \quad Sdf(2n) = 2^k \cdot Sdf(n).
\]
Since \( Sdf(2n) \leq 2n \), we see from (5.13) that \( k=1 \) and
\[
(5.14) \quad Sdf(2n) = 2 \cdot Sdf(n).
\]
Notice that (5.14) holds for any odd integer \( n \) with \( n \geq 1 \). We get (iii).

If \( 2|m \) and \( 2 \mid n \), then we have
\[
(5.15) \quad Sdf(mn) \leq 2 \cdot Sdf(m) + Sdf(n).
\]
By (5.3) and (5.15), we get
\[
(5.16) \quad 2m \geq 2 \cdot Sdf(m) \geq (m^k - 1) \cdot Sdf(n).
\]
When \( n=2 \), we see from (5.16) that \( m=3 \) and \( k=1 \). When \( n>2 \), we get from (5.16) that \( 2m \geq 4(m^k - 1) \geq 4(m - 1) > 2m \), a contradiction.
If $2 \mid m$ and $2 \mid n$, then we have

\begin{equation}
Sdfs(mn) \leq 2 \cdot Sdfs(m) + 2 \cdot Sdfs(n) - 1.
\end{equation}

By (5.3) and (5.17), we get

\begin{equation}
2m - 1 \geq 2 \cdot Sdfs(m) - 1 \geq (m^k - 2) \cdot Sdfs(n) \geq 3(m^k - 2).
\end{equation}

It implies that $k = 1$ and $m = 3$ or 5. When $m = 3$ and $k = 1$, we get from (5.3) that

\begin{equation}
Sdfs(3n) = 3 \cdot Sdfs(n).
\end{equation}

Since $Sdfs(3n) \leq Sdfs(n) + 6$, we find from (5.19) that $n = 3$. Thus, we get (iv). When $m = 5$ and $k = 1$, we have

\begin{equation}
Sdfs(5n) = 5 \cdot Sdfs(n).
\end{equation}

Since $Sdfs(5n) \leq Sdfs(n) + 10$, (5.20) is impossible. To sum up, the theorem is proved.

Let $p$ be a prime, and let $N(p)$ denote the number of solutions $x$ of the equation

\begin{equation}
Sdfs(x) = p, \ x \in \mathbb{N}.
\end{equation}

Recently, Johnson showed that if $p$ is an odd prime, then

\begin{equation}
N(p) = 2 \cdot \sigma^2.
\end{equation}

Unfortunately, the above-mentioned result is false. For example, by (5.22), we get $N(19) = 2^4 = 256$. However, the fact is that $N(19) = 240$. We now give a general result as follows.

**Theorem 5.4.** For any positive integer $t$, let $p(t)$ denote the $t$th odd prime. If $p = p(t)$, then
(5.23) \[ N(p) = \prod_{i=1}^{t-1} (a(i) + 1), \]

where

(5.24) \[ a(i) = \prod_{m=1}^{(p-2)/2} \left( \left\lfloor \frac{p-2}{(p(i))^{m}} \right\rfloor - \left\lfloor \frac{(p-3)/2}{(p(i))^{m}} \right\rfloor \right), i = 1, 2, \ldots, t - 1. \]

Proof. Let \( x \) be a solution of (5.21). It is an obvious fact that

(5.25) \[ x = dp, \]

where \( d \) is a divisor of \((p-2)!!\). So we have

(5.26) \[ N(p) = d((p-2)!!), \]

where \( d((p-2)!!) \) is the number of distinct divisors \( d \) of \((p-2)!!\).

By the definition of \((p-2)!!\), we have

(5.27) \[ (p-2)!! = (p(1))^{a(1)}(p(2))^{a(2)} \cdots (p(t-1))^{a(t-1)}, \]

where

(5.28) \[ a(i) = \text{ord}(p(i), (p-2)!!), \quad i = 1, 2, \ldots, t-1. \]

Notice that

(5.29) \[ (p-2)!! = \frac{(p-2)!}{2^{(p-3)/2} \left( \frac{p-3}{2} \right)}. \]

We get

(5.30) \[ \text{ord}(p(i), (p-2)!!) = \text{ord}(p(i), (p-2)!!) - \text{ord} \left( p(i), \left( \frac{p-3}{2} \right) ! \right). \]

Therefore, by Theorem 1.11.1 of [3], we see from (5.28) and (5.30) that \( a(i) \) \((i = 1, 2, \ldots, t-1)\) satisfy (5.24). Further, by Theorem 273 of [2], we get from (5.27) that
Thus, by (5.26), we obtain (5.23). The theorem is proved.

References