ON THE SUMATORY FUNCTION ASSOCIATED TO

THE SMARANDACHE FUNCTION

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It is said that for every numerical function \( f \) it can be attached the sumatory function:

\[
F(n) = \sum_{d \mid n} f(d)
\]

The function \( f \) is expressed as:

\[
f(n) = \sum_{\mu(v) \cdot F(v)} \mu(u)
\]

Where \( \mu \) is the Möbius function \( (\mu(1) = 1, \mu(n) = 0 \text{ if } n \text{ is divisible by the square of a prime number, } \mu(n) = (-1)^k \text{ if } n \text{ is the product of } k \text{ different prime numbers}) \)

If \( f \) is the Smarandache function and \( n = p^a \) then:

\[
F_s(p^a) = \sum_{j=1}^{a} S(p^j)
\]

In [2] it is proved that

\[
S(p^j) = p - 1 \cdot j - \alpha_{p_1}(j)
\]

where \( \alpha_{p_1}(j) \) is the sum of the digits of the integer \( j \), written in the generalised scale

\[
[p] = a_1(p), a_2(p), \ldots, a_\chi(p), \ldots
\]

with \( a_\chi(p) = (p^n - 1)/(p - 1) \)

So

\[
F_s(p^a) = \sum_{j=1}^{a} S(p^j) = (p - 1) \frac{a(a + 1)}{2} + \sum_{j=1}^{a} \alpha_{p_1}(j)
\]

Using the expression of \( \alpha \) given by (3) it results

\[
(a + 1)(S(p^a) - \alpha_{p_1}(a)) = 2(F_s(p^a) - \sum_{j=1}^{a} \alpha_{p_1}(j))
\]

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In the following we give an algorithm to calculate the sum in the right hand of (4). For this, let \( a_{i} = k_{s} \cdot k_{s-1} \cdot \ldots \cdot k_{t} \) the expression of \( a \) in the scale \([p]\) and \( j_{i} = k_{s} \cdot k_{s-1} \cdot \ldots \cdot k_{t} \). We shall say that \( i \) are the digits of order \( i \), for \( j = 1, 2, \ldots, \alpha \).

To calculate the sum of all the digits of order \( i \), let \( \omega_{i} = \alpha - a_{i} + 1 \).

Now we consider two cases:

(i) if \( k_{i} \neq 0 \), let:

\[
z_{i}(a) = (k_{s} \cdot k_{s-1} \cdots k_{t})_{\alpha \cap (p)}; \text{ the equality } u = a_{i}(p) \text{ denoting that for the number written between parantheses, the classe of units is } a_{i}(p).
\]

Then \( z_{i}(a) \) is the number of all zeros of order \( i \) for the integers \( j \leq \alpha \) and \( \omega_{i} = \omega_{i}(a) - z_{i}(a) \) is the number of the non-null digits.

(ii) if \( k_{i} = 0 \), let \( \beta \) the greatest number, less than \( \alpha \), having a non-null digit of order \( i \). Then \( \beta \) is of the form:

\[
\beta_{(p)} = k_{s} \cdot k_{s-1} \cdots k_{t-1} \cdot (k_{t-1} - 1)p00 \ldots 0
\]

and of course \( s_{i}(\alpha) = s_{i}(\beta) \). It results that there exist \( a_{i}(\beta) \) non-null digits of order \( i \).

Let \( A_{i}, B_{i}, R_{i}, P_{i} \) given by equalities:

\[
A_{i} = a_{i}((p - 1)a_{i}(p) + 1) + R_{i} = a_{i}(a_{i}(p) - a_{i}(p)) + R_{i};
\]

\[
R_{i} = B_{i}a_{i}(p) + P_{i}.
\]

Then

\[
s_{i}(a) = A_{i}a_{i}(p) \frac{p(p - 1)}{z} - A_{i}p + a_{i}(p) \frac{p(p - 1)}{z} + P_{i}(B_{i} + 1)
\]

and

\[
\sum_{p, j = 1}^{\alpha} s_{i}(a) = \sum_{i = 1}^{\alpha} A_{i}a_{i}(p) + p \sum_{i = 1}^{\alpha} A_{i} + \frac{1}{z} \sum_{i = 1}^{\alpha} A_{i}a_{i}(p)B_{i}(B_{i} + 1) + \sum_{i = 1}^{\alpha} A_{i}(B_{i} + 1)
\]

For example if \( \alpha = 149 \) and \( p = 3 \) it results:

\[
[3] 1, 4, 13, 40, 121, \ldots
\]
\[ x_{10} = 10202 \, , \, \nu_1(\alpha) = (1020)_{\alpha_1(3)} = 48 \, , \, \alpha_1 = \nu_1(\alpha) - x_1(\alpha) = 101 \]

For \( \beta_{10} = 10130 = 146 \) it results \( \nu_2(\beta) = 143 \), \( z_2(\beta) = \)
\( (101)_{\alpha_2(3)} = u_3 + u = 3u_2 + 1 + u = 3(3u + 1) + 1 + u = 44 \)
\( x_2 = 99 \, , \, \nu_3(\alpha) = 137 \, , \, z_3(\alpha) = (10)_{\alpha_3(3)} = 40 \, , \, \alpha_3 = 97 \).

For \( \beta_{30} = 3000 = 120 \) it results \( \nu_3(\beta) = 81 \), \( z_4(\beta) = 0 \), \( \alpha_4 = 108 \).

\( \nu_3(\alpha) = 29 \), \( z_5(\alpha) = 0 \), \( \alpha_5 = 29 \), and

\[ A_1 = \left[ \frac{\alpha_1}{x_2 - 3} \right] = 33 \, , \, B_1 = \left[ \frac{2}{\alpha_1} \right] \, , \, \beta_1 = 0 \, , \, s_1 = 201 \]

Analogously \( s_2 = 165 \, , \, s_3 = 145 \, , \, s_4 = 123 \) and \( s_5 = 129 \), so
\[ \sum_{i=1}^{149} \alpha_1(i) = 633 \, , \, F_s(3^{149}) = 22983 \, . \]

Now let us consider \( n = P_1P_2 \ldots P_k \), with \( p_1 < P_2 < \ldots < P_k \) prime numbers. Of course, \( S(n) = p_k \) and from
\[ F_s(1) = S(1) = 0 \]
\[ F_s(P_1) = S(1) + S(P_1) = p_1 \]
\[ F_s(P_1P_2) = P_1 + 2P_2 = F(P_1) + 2P_2 \]
\[ F_s(P_1P_2P_3) = P_1 + 2P_2 + 2^2P_3 = F(P_1P_2) + 2^2P_3 \]

it results:
\[ F_s(P_1P_2 \ldots P_k) = F(P_1P_2 \ldots P_{k-1}) + 2^{k-1}P_k \]

That is:
\[ F(P_1P_2 \ldots P_k) = \sum_{i=1}^{k} 2^{i-1}P_i \]

The equality (2) becomes:
\[ P_k = S(n) = \sum_{n \neq n} \mu(n) F_s(v) = \]
\[ = F(n) - \sum_{i=1}^{k} F(\frac{n}{P_i}) + \sum_{i,j} F(\frac{n}{P_iP_j}) + \ldots + \sum_{i=1}^{k} F(P_i) \]

and became \( F(p_i) = p_i \), it results:

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\[ F(\frac{n}{p_i}) = F(p_i \cdot p_2 \cdots p_{i-1} \cdot p_{i+1} \cdots p_k) = \sum_{j=1}^{i-1} 2^{-j} p_j + \sum_{j=i+1}^{k} 2^{-j} p_j = F(p_i \cdot p_2 \cdots p_{i-1}) + 2^{-i} F(p_{i+1} \cdot p_{i+2} \cdots p_k).\]

Analogously,
\[ F(\frac{n}{p_i \cdot p_j}) = F(p_i \cdot p_2 \cdots p_{i-1} \cdot p_{j+1} \cdots p_k) + 2^{-i} F(p_{i+1} \cdot p_{i+2} \cdots p_k).\]

Finally, we point out as an open problem that, by the Shapiro's theorem, if it exist a numerical function \( g : \mathbb{N} \rightarrow \mathbb{R} \) such that
\[ g(n) = \sum_{d \mid n} p(n) S(\frac{n}{d}) \]
were \( p \) is a totally multiplicative function and \( p(1) = 1 \), then
\[ S(n) = \sum_{d \mid n} \mu(d) P(d) g(\frac{n}{d}) \]

REFERENCES


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