Perfect Powers in Smarandache Type Expressions

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In [2] and [3] the authors ask how many primes are of the Smarandache form (see [10]) $x^y + y^z$, where $\gcd(x, y) = 1$ and $x, y \geq 2$. In [6] the author showed that there are only finitely many numbers of the above form which are products of factorials.

In this article we propose the following

Conjecture 1. Let $a$, $b$, and $c$ be three integers with $ab \neq 0$. Then the equation

$$ax^y + by^z = cz^n$$

with $x, y, n \geq 2$, and $\gcd(x, y) = 1$, (1)

has finitely many solutions $(x, y, z, n)$.

We announce the following result:

Theorem 1. The "abc Conjecture" implies Conjecture 1.

The proof of Theorem 1 is based on an idea of Lang (see [5]).

For any integer $k$ let $P(k)$ be the largest prime number dividing $k$ with the convention that $P(0) = P(\pm 1) = 1$. We have the following result.

Theorem 2. Let $a$, $b$, and $c$ be three integers with $ab \neq 0$. Let $P > 0$ be a fixed positive integer. Then the equation

$$ax^y + by^z = cz^n$$

with $x, y, n \geq 2$, $\gcd(x, y) = 1$, and $P(y) < P$, (2)

has finitely many solutions $(x, y, z, n)$. Moreover, there exists a computable positive number $C$ depending only on $a$, $b$, $c$, and $P$ such that all the solutions of equation (2) satisfy $\max(x, y) < C$.

The proof of theorem 2 uses lower bounds for linear forms in logarithms of algebraic numbers.

Conjecture 2. The only solutions of the equation

$$x^y \pm y^z = z^n$$

with $x, y, n \geq 2$, $z > 0$, $\gcd(x, y) = 1$, (3)

are $(x, y, z, n) = (3, 2, 1, n)$.

We have the following results:

Theorem 3. The equation

$$x^y \pm y^z = z^2$$

with $x, y \geq 2$, and $\gcd(x, y) = 1$, (4)

has finitely many solutions $(x, y, z)$ with $2 \mid xy$. Moreover, all such solutions satisfy $\max(x, y) < 3 \cdot 10^{143}$. 
The proof of Theorem 3 uses lower bounds for linear forms in logarithms of algebraic numbers.

Theorem 4. The equation

\[ 2^y + y^2 = z^n \]  

has no solutions \((y, z, n)\) such that \(y\) is odd and \(n > 1\).

The proof of theorem 4 is elementary and uses the fact that \(\mathbb{Z}[i\sqrt{2}]\) is an UFD.

2. Preliminary Results

We begin by stating the \(abc\) Conjecture as it appears in [5]. Let \(k\) be a nonzero integer. Define the radical of \(k\) to be

\[ N_0(k) = \prod_{p \mid k} p \]  

i.e. the product of the distinct primes dividing \(k\). Notice that if \(x\) and \(y\) are integers, then

\[ N_0(xy) = N_0(x)N_0(y), \]

and if \(\gcd(x, y) = 1\), then

\[ N_0(xy) = N_0(x)N_0(y). \]

The \(abc\) Conjecture ([5]). Given \(\epsilon > 0\) there exists a number \(C(\epsilon)\) having the following property. For any nonzero relatively prime integers \(a, b, c\) such that \(a + b = c\) we have

\[ \max(|a|, |b|, |c|) < C(\epsilon)N_0(abc)^{1+\epsilon}. \]

The proofs of theorems 2 and 3 use estimations of linear forms in logarithms of algebraic numbers.

Suppose that \(\zeta_1, \ldots, \zeta_l\) are algebraic numbers, not 0 or 1, of heights not exceeding \(A_1, \ldots, A_l\), respectively. We assume \(A_m \geq e^e\) for \(m = 1, \ldots, l\). Put \(\Omega = \log A_1 \ldots \log A_l\). Let \(F = \mathbb{Q}[\zeta_1, \ldots, \zeta_l]\). Let \(n_1, \ldots, n_l\) be integers, not all 0, and let \(B \geq \max |n_m|\). We assume \(B \geq e^2\). The following result is due to Baker and Wüstholz.

Theorem BW ([1]). If \(\zeta_1^{n_1} \ldots \zeta_l^{n_l} \neq 1\), then

\[ |\zeta_1^{n_1} \ldots \zeta_l^{n_l} - 1| > \frac{1}{2} \exp\left(-\frac{16(l + 1)A}{\log B}\right). \]  

In fact, Baker and Wüstholz showed that if \(\log \zeta_1, \ldots, \log \zeta_l\) are any fixed values of the logarithms, and \(\Lambda = n_1 \log \zeta_1 + \ldots + n_l \log \zeta_l \neq 0\), then

\[ \log |\Lambda| > -(16l\log B)_{2(l+2)} \Omega \log B. \]  

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Now (7) follows easily from (8) via an argument similar to the one used by Shorey et al. in their paper [9].

We also need the following p-adic analogue of theorem BW which is due to Alf van der Poorten.

**Theorem vdP ([7])**. Let π be a prime ideal of $\mathbb{F}$ lying above a prime integer $p$. Then,

$$\text{ord}_π(ζ_1^n \ldots ζ_l^n - 1) < (16(l + 1)dπ)^{12(l+1)} \frac{p^{dπ}}{\log p} Ω(\log B)^2.$$  \hspace{1cm} (9)

We also need the following two results.

**Theorem K ([4])**. Let $A$ and $B$ be nonzero rational integers. Let $m ≥ 2$ and $n ≥ 2$ with $mn ≥ 6$ be rational integers. For any two integers $x$ and $y$ let $X = \max(|x|, |y|)$. Then

$$P(Ax^m + By^n) > C(\log_2 X \log_3 X)^{1/2}$$  \hspace{1cm} (10)

where $C > 0$ is a computable constant depending only on $A$, $B$, $m$ and $n$.

**Theorem S ([8])**. Let $n > 1$ and $A$, $B$ be nonzero integers. For integers $m > 3$, $x$ and $y$ with $|x| > 1$, $\gcd(x, y) = 1$, and $Ax^m + By^n ≠ 0$, we have

$$P(Ax^m + By^n) ≥ C((\log m)(\log \log m))^{1/2}$$  \hspace{1cm} (11)

and

$$|Ax^m + By^n| ≥ \exp \left( C((\log m)(\log \log m))^{1/2} \right)$$  \hspace{1cm} (12)

where $C > 0$ is a computable number depending only on $A$, $B$ and $n$.

Let $K$ be a finite extension of $\mathbb{Q}$ of degree $d$, and let $𝒪_K$ be the ring of algebraic integers inside $K$. For any element $γ ∈ 𝒪_K$, let $[γ]$ be the ideal generated by $γ$ in $𝒪_K$. For any ideal $I$ in $𝒪_K$, let $N(I)$ be the norm of $I$. Let $π_1$, $π_2$, ..., $π_l$ be a set of prime ideals in $𝒪_K$. Put

$$p = \max P(N(π_i)).$$

Write

$$π_i^h = [p_i] \quad \text{for } i = 1, ..., l$$

where $p_1$, $p_2$, ..., $p_l ∈ 𝒪_K$ and $h$ is the class number of $K$. Denote by $S$ the set of all elements $α$ of $𝒪_K$ such that $[α]$ is exclusively composed of prime ideals $π_1$, $π_2$, ..., $π_l$. Then we have
Lemma T. ([9]). Let $\alpha \in S$. Assume that

$$[\alpha] = a_1^{b_1} \pi_2^{b_2} \ldots \pi_i^{b_i}.$$  

There exist $\beta \in O_K$ with $|N(\beta)| \leq p^{\text{dhl}}$ and a unit $\epsilon \in O_K$ such that

$$\alpha = \epsilon \beta p_1^{a_1^1} p_2^{a_2} \ldots p_i^{a_i}.$$  

Moreover,

$$b_i = a_i / h + c_i$$  

for some $0 \leq c_i < h$.

3. The Proofs

The Proof of Theorem 1. We may assume that $\gcd(a, b, c) = 1$. By $C_1$, $C_2$, ..., we shall denote computable positive numbers depending only on $a$, $b$, $c$. Let $(x, y, z, n)$ be a solution of (1). Assume that $x > y$, and that $x > 3$. Let $d = \gcd(ax, by)$. Notice that $d | ab$. Equation (1) becomes

$$\frac{ax}{d} + \frac{by}{d} = \frac{cz^n}{d}.$$  

By the abc Conjecture for $\epsilon = 2/3$ it follows that

$$\max (|ax|, |by|, |cz^n|) < \frac{C(2/3)N_0(abc)^{5/3}}{d^2} N_0(xyz)^{5/3}.$$  

Let

$$C_1 = C(2/3)N_0(abc)^{5/3}.$$  

Since $d \geq 1$, and $|b| \geq 1$, from inequality (14) it follows that

$$y^x \leq |by|^z < C_1(xyz)^{5/3} < C_1x^{10/3}|z|^{5/3}.$$  

Since $x > \min (y, 3)$, it follows easily that $y^x > x^y$. Hence,

$$|z|^n = |a \frac{x}{c} + b \frac{y}{c}| < C_2 y^x$$

where $C_2 = \frac{|a| + |b|}{|c|}$. We conclude that

$$|z| < C_2^{1/n} y^{x/n} \leq C_2^{1/2} y^{x/n}.$$  

(16)

Combining inequalities (15) and (16) it follows that

$$y^x < C_1 C_2^{5/6} x^{10/3} y^{(5z/3n)},$$

or

$$y^{x(1-5/3n)} < C_3 x^{10/3},$$  

(17)
where \( C_3 = C_1 C_2^{5/6} \). Since \( 2 \leq y \) and \( 2 \leq n \), it follows that
\[
2^{5/6} \leq 2^{(1-5/3n)} < C_3 n^{10/3}.
\] (18)

Inequality (18) clearly shows that \( x < C_4 \).

**The Proof of Theorem 2.** We may assume that
\[
P \geq \max (P(a), P(b), P(c)).
\]

By \( C_1, C_2, \ldots \), we shall denote computable positive numbers depending only on \( a, b, c, P \). We begin by showing that \( n \) is bounded. Fix \( d \in \{2, 3, \ldots, P-1\} \). Suppose that \( x, y, z, n \) is a solution of (2) with \( n > 3 \) and \( d \mid y \). Since
\[
y x = cz^n - a \left( x^{y/d} \right)^d
\]
(19)
it follows, by Theorem S, that
\[
P = P \left( y x \right) = P \left( cz^n - a \left( x^{y/d} \right)^d \right) > C_1 \left( (\log n)(\log \log n) \right)^{1/2}
\]
(20)

where \( C_1 \) is a computable number depending only on \( a, c, d \). Inequality (20) shows that \( n < C_2 \).

Suppose now that \( ny \geq 6 \). Let \( X = \max (x, |z|) \). From equation (19) and theorem K, it follows that
\[
P = P \left( y x \right) = P \left( cz^n - a x^y \right) > C_3 \left( \log_2 X \log_3 X \right)^{1/2},
\]
(21)

where \( C_3 > 0 \) is a computable constant depending only on \( a, c, \) and \( C_2 \). From inequality (21) it follows that \( X < C_3 \). Let \( C_4 = \max (C_2, C_3) \). It follows that, if \( ny \geq 6 \), then \( \max (x, |z|, n) < C_4 \). We now show that \( y \) is bounded as well. Suppose that \( y > \max (C_4, e^2) \). Rewrite equation (2) as
\[
\left| \frac{cz^n}{a x^y} = 1 - \left( \frac{-b}{a} \right) y x^{-y} \right|.
\]
(22)

Let \( A > e^a \) be an upper bound for the height of \( -b/a \) and \( C_4 \). Let \( \Omega = (\log A)^3 \). From theorem BW we conclude that
\[
\log |c| + n \log |z| - \log |a| - y \log x > - \log 2 - 64^{12} \Omega \log y.
\]
(23)

Since \( x \geq 2 \), and \( \max (x, |z|, n) < C_4 \), it follows, by inequality (23), that
\[
y \log 2 - 64^{12} \Omega \log y \leq y \log x - 64^{12} \Omega \log y < C_4 \log C_4 - \log |a| + \log |c| + \log 2.
\]
(24)

From equation (24) it follows that \( y < C_5 \).
Suppose now that \( n = y = 2 \). We first bound \( z \) in terms of \( x \). Rewrite equation (2) as

\[
z^2 = 2^x \left| \frac{b}{c^2} \right| 1 + \left( \frac{a}{b} \right) \left( \frac{x^2}{2^x} \right).
\]  
(25)

Let \( C_6 > 0 \) be a computable positive number depending only on \( a \) and \( b \) such that

\[
\left| \frac{a}{b} \left( \frac{x^2}{2^x} \right) \right| < \frac{1}{2} \quad \text{for } x > C_6.
\]  
(26)

From equation (25) and inequality (26), it follows that

\[
2^x \left| \frac{b}{2c} \right| < 2^x \left| \frac{b}{c} \right| \left( 1 - \left| \frac{a}{b} \left( \frac{x^2}{2^x} \right) \right| \right) < x^2 < 2^x \left| \frac{b}{c} \right| \left( 1 + \left| \frac{a}{b} \left( \frac{x^2}{2^x} \right) \right| \right) < 2^x \left| \frac{b}{2c} \right|
\]  
(27)

for \( x > C_6 \). Taking logarithms in inequality (27) we obtain

\[
xC_7 + C_8 < \log z < xC_7 + C_9 \quad \text{for } x > C_6
\]  
(28)

where \( C_7 = \log \frac{2}{x}, C_8 = \frac{\log |b| - \log 2|c|}{2}, \) and \( C_9 = \frac{\log |3b| - \log |2c|}{2} \). We now rewrite equation (2) as

\[
(cx)^2 - acc^2 = ab2^x.
\]  
(29)

Let \( \alpha = \sqrt{ac} \). Then

\[
(cx + \alpha x)(cx - \alpha x) = ab2^x.
\]  
(30)

We distinguish 2 cases.

CASE 1. \( ac < 0 \). Let \( K = \mathbb{Q}(\alpha) \). Since \( ac < 0 \), it follows that all the units of \( \mathcal{O}_K \) are roots of unity. Since \( K \) is a quadratic field, it follows that the ideal \([2]\) has at most two prime divisors. Since

\[
\gcd \left( \left| cz + \alpha x \right|, \left| cz - \alpha x \right| \right) \bigg| 2|abc|
\]

it follows, by lemma T, that

\[
 cz + \alpha x = e\beta p^u
\]  
(31)

where \( \frac{x}{h} - 1 < u \leq \frac{x}{h} \), and \( e, \beta, p \in \mathcal{O}_K \) are such that \( |e| = 1, |p| = 2^{h/2} \), where \( h \) is the class number of \( K \), and \( |\beta| < C_{10} \) where \( C_{10} \) is a computable number depending only on \( a, b, \) and \( c \). Conjugating equation (31) we get

\[
 cz - \alpha x = \overline{e}\beta p^u.
\]  
(32)

From equations (31) and (32) it follows that

\[
2\alpha x = e\beta p^u \left( 1 - (-e^{-2})(\beta)^{-1}\overline{\beta}(p)^{-u}(\overline{p})^u \right).
\]

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Hence,

$$2|\alpha|x = |\beta||p|^u|1 - (-e^{-2})^{-1}B(p)^{-u}(p)^u|$$  \hspace{1cm} (33)

Taking logarithms in equation (33) we obtain

$$\log(2|\alpha|) + \log x = \log |\beta| + u \log p + \log |1 - (-e^{-2})^{-1}B(p)^{-u}(p)^u|.$$  \hspace{1cm} (34)

Let $A$, and $P$ be upper bounds for the heights of $-e^{-2}B^{-1}$ and $p$, respectively. Assume that $\min(A, P) > e^a$. Let $\Omega = \log A(\log P)^2$. Assume also that $\frac{x}{h} > 1 + e^2$. From equation (34), theorem BW, the fact that $|p| = 2^{h/2}$, and the fact that $\frac{x}{h} - 1 < u \leq \frac{x}{h}$, we obtain that

$$\log(2|\alpha|) + \log x > \log |\beta| + u \log |p| - \log 2 - 64^{12}\Omega \log u >$$

$$\log |\beta| + \left(\frac{x}{h} - 1\right) \cdot \left(\frac{h}{2}\right) \log 2 - \log 2 - 64^{12}\Omega \log(x/h).$$  \hspace{1cm} (35)

Inequality (35) clearly shows that $x < C_{11}$.

CASE 2. $ac > 0$. We may assume that both $a$ and $c$ are positive. If $b < 0$, equation (2) can be rewritten as

$$|a|x^2 - |b|2^x = |c|z^2 > 0$$  \hspace{1cm} (36)

Equation (36) clearly shows that $x < C_{12}$. Hence, we assume that $b > 0$. We distinguish two subcases.

CASE 2.1. $\sqrt{ac} \in \mathbb{Z}$. In this case, from equation

$$(c|z| + ax)(c|z| - ax) = bc2^x$$

and from the fact that

$$\gcd\left(c|z| + ax, c|z| - ax\right) | 2\alpha cb$$  \hspace{1cm} (37)

it follows easily that

$$\left\{
\begin{array}{l}
|c|z| + ax = \beta 2^u \\
|c|z| - ax = \gamma
\end{array}\right.$$  \hspace{1cm} (38)

where $\beta$, $\gamma$, $u$ are positive integers with $0 < \beta < bc$, $\gamma < (bc) \cdot (2\alpha cb)$ and $u > x - \text{ord}_2(2\alpha cb)$. From equation (38) it follows that

$$2\alpha x = \beta 2^x - \gamma.$$  \hspace{1cm} (39)

From equation (39), and from the fact that $0 < \beta < bc$, $\gamma < (bc) \cdot (2\alpha cb)$, and $u > x - \text{ord}_2(2\alpha cb)$, it follows that $x < C_{13}$.
CASE 2.2. \( \sqrt{ac} \not\in \mathbb{Z} \). Let \( K = \mathbb{Q}[\alpha] \). Let \( \varepsilon \) be a generator of the torsion free subgroup of the units group of \( \mathcal{O}_K \). From equation (37) and lemma T, it follows that

\[
c[z] + \alpha x = \varepsilon^m \beta_1 p_1^x
\]

where \( \frac{x}{h} - 1 < u \leq \frac{x}{h} \), and \( \beta, p_1 \in \mathcal{O}_K \) are such that \( 1 < \beta_1 < C_{14} \) for some computable constant \( C_{14} \), and \( 1 < p_1 < 2^h \cdot \varepsilon \). From equation (40), it follows that

\[
c[z] - \alpha x = \varepsilon^{-m} \beta_2 p_2^x
\]

where \( \beta_2 = |\beta_1|^2 / \beta_1 \), and \( p_2 = 2^h / p_1 \). Suppose now that \( x > C_\delta \). Since

\[
e^m = p_1^{-u} \beta_1^{-1}(c[z] + \alpha x)
\]

it follows, from inequality (28), and from the fact that \( \frac{x}{h} - 1 < u \leq \frac{x}{h} \) and \( 1 < p_1 < 2^h \cdot \varepsilon \), that

\[
|m| < C_{13} x + C_{16} \quad \text{for } x > C_\delta,
\]

for some computable constants \( C_{15} \) and \( C_{18} \) depending only on \( a, b, \) and \( c \). From equations (40) and (41), it follows that

\[
2\alpha x = e^m \beta_1 p_1^x \cdot \left(1 - e^{-2m(1)(-1)\beta_2(p_1)^{-u}p_2^u}\right)
\]

or

\[
2\alpha x = (c[z] + \alpha x) \cdot \left(1 - e^{-2m(1)(-1)\beta_2(p_1)^{-u}p_2^u}\right).
\]

Let \( A_1, A_2, A_3, A_4 \) be upper bounds for the heights of \( \varepsilon, (\beta_1)^{-1}\beta_2, p_1, p_2 \) respectively. Assume that \( \min(A_1, A_2, A_3, A_4) > e^\varepsilon \). Denote \( \Omega = \prod_{i=1}^4 \log A_i \). Denote \( C_{17} = \max(2C_{15}, 1/h) \). From inequality (42), it follows that

\[
\max(2|m|, u) < C_{17} x + C_{16}.
\]

Let \( B = C_{17} x + C_{16} \). Taking logarithms in equation (43), and applying theorem BW, we obtain

\[
\log(2\alpha) + \log x = \log(c[z] + \alpha x) + \log\left|1 - e^{-2m(1)(-1)\beta_2(p_1)^{-u}p_2^u}\right| >
\]

\[
\log(c[z] + \alpha x) - \log 2 - 80^{14} \Omega \log(C_{17} x + C_{16}).
\]

Combining inequalities (28) and (45) we obtain

\[
\log(4\alpha) + \log x + 80^{14} \Omega \log(C_{17} x + C_{16}) > \log(c[z] + \alpha x) > \log z > C_7 x + C_8
\]

This last inequality clearly shows that \( x < C_{18} \).
The Proof of Theorem 3. We treat only the equation
\[x^y + y^x = z^2.\]

We may assume that \( z = 0 \). First notice that, since \( \gcd(x, y) = 1 \), it follows that \( \gcd(x, z) = \gcd(y, z) = 1 \). Rewrite equation (4) as
\[x^y = (z + y^z/2)(z - y^z/2).\]

Since \( \gcd(z, y^z/2) = 1 \) and both \( x \) and \( y \) are odd, it follows that
\[\gcd(z + y^z/2, z - y^z/2) = 2.\]

Write \( x = 2d_1d_2 \) such that either one of the following holds
\[\begin{align*}
\{ z + y^z/2 = 2^{v-1}d_1^y \quad \text{or} \quad z - y^z/2 = 2d_2^y \\
\quad \text{or} \quad z - y^z/2 = 2^{v-1}d_1^y
\end{align*}\]

Hence, either
\[y^z/2 = 2^{v-2}d_1^y - d_2^y \quad \text{(47)}\]
or
\[y^z/2 = d_1^y - 2^{v-2}d_2^y \quad \text{(48)}\]

We proceed in several steps.

Step 1. (1) If \( x > y \) then either \( y \leq 9 \) and \( x < 27 \), or \( y > 9 \) and \( x < 3y \).

(2) If \( x < y \) and \( y > 2.6 \cdot 10^{21} \), then \( y < 4x \).

(1) Assume first that \( x > y \). Since
\[y^z/2 = 2^{v-2}d_1^y - d_2^y \quad \text{or} \quad y^z/2 = d_1^y - 2^{v-2}d_2^y\]
it follows that
\[y^z/2 < 2^{v-1}d_1^y < (2d_1)^y < x^y \quad \text{or} \quad y^z/2 < d_1^y < x^y. \quad \text{(49)}\]

Hence,
\[\frac{x}{\log y} < y \log x. \quad \text{(50)}\]

Inequality (50) is equivalent to
\[\frac{x}{\log x} < 2 \frac{y}{\log y}. \quad \text{(51)}\]

If \( y \leq 9 \), then one can check easily that (51) implies \( x < 27 \). Suppose now that \( y > 9 \). We show that inequality (51) implies \( x < 3y \). Indeed, assume that \( x \geq 3y \). Then
\[\frac{3y}{\log 3 + \log y} = \frac{3y}{\log(3y)} \leq \frac{x}{\log x} < \frac{2y}{\log y}. \quad \text{(52)}\]
Inequality (52) is equivalent to

\[ 3 \log y < \log 9 + 2 \log y \]

or \( y < 9 \). This contradiction shows that \( x < 3y \) for \( y > 9 \).

(2) Assume now that \( x < y \). Suppose first that

\[ y^{x/2} = 2^{y-2}d_1^y - d_2^y. \]

In this case

\[ (2d_1)^y > 2^{y-2}d_1^y = d_2^y + y^{x/2} > d_2^y \]

therefore \( 2d_1 > d_2 \). Since \( x = 2d_1d_2 \), it follows that \( 2d_1 > \sqrt{x} \), or \( d_1 > \frac{\sqrt{x}}{2} \).

Suppose now that

\[ y^{x/2} = d_1^y - 2^{y-2}d_2^y. \]

In this case,

\[ d_1^y > 2^{y-2}d_2^y > d_2^y \]

or \( d_1 > d_2 \). We obtain that \( d_1 > \sqrt{d_1d_2} = \frac{\sqrt{x}}{2} > \frac{\sqrt{x}}{2} \).

If equality (47) holds, it follows that

\[ y^{x/2} = 2^{y-2}d_1^y \mid 1 - 2^{-y-2} \left( \frac{d_1}{d_1} \right)^y \mid \geq d_1^y \mid 1 - 2^{-y-2} \left( \frac{d_2}{d_1} \right)^y \mid. \] (53)

On the other hand, if equality (48) holds, then

\[ y^{x/2} = d_1^y \mid 1 - 2^{-(y-2)} \left( \frac{d_2}{d_1} \right)^y \mid. \] (54)

From inequality (53) and equation (54), we conclude that, in either case,

\[ y^{x/2} \geq d_1^y \mid 1 - 2^{-(y-2)} \left( \frac{d_2}{d_1} \right)^y \mid \] (55)

for some \( \epsilon \in \{ \pm 1 \} \). Suppose now that \( x > e^x \). By theorem BW, and inequality (55), it follows that

\[ \frac{x}{2} \log y \geq y \log d_1 - \log 2 - 48^{10}e \log x \log y \geq \]

\[ y \log \frac{\sqrt{x}}{2} - \log 2 - 48^{10}e \log x \log y \] (56)

or

\[ 48^{10}e \log x \log y + \log 2 + \frac{x}{2} \log y > y \log \frac{\sqrt{x}}{2}. \] (57)
CASE 1. Assume that $x < 2^6$. From inequality (57), it follows that

$$48^{10} e \cdot 6 \log 2 \cdot \log y + \log 2 + 2^5 \log y > y \log \frac{\sqrt{e} x}{2} > \frac{y}{2}$$

or

$$(48^{10} e \cdot 6 \log 2 + 2^5) \log y + \log 2 > \frac{y}{2}$$

or

$$2(48^{10} e \cdot 6 \log 2 + 2^5 + 1) > \frac{y}{\log y}. \quad (58)$$

Let $C_1 = 2(48^{10} e \cdot 6 \log 2 + 2^5 + 1)$. From inequality (58) and lemma 2 in [6], it follows that

$$y < C_1 \log^2 C_1 < 2(48^{10} e \cdot 6 \log 2 + 2^5 + 1) \cdot 42^2 < 2.6 \cdot 10^{21}. \quad (59)$$

CASE 2. Assume that $x \geq 2^6$. Then,

$$d_1 > \frac{\sqrt{x}}{2} \geq 3 \sqrt{x}.$$  

Inequality (56) becomes

$$48^{10} e \log x \log y + \log 2 + \frac{x}{2} \log y > \frac{1}{3} y \log x$$

or

$$3e48^{10} \log x \log y + \log 3 + \frac{3}{2} x \log y > y \log x$$

or

$$(3e48^{10} + 1) \log x \log y + \frac{3}{2} x \log y > y \log x$$

or

$$3e48^{10} + 1 + \frac{3}{2} \frac{x}{\log x} > \frac{y}{\log y}. \quad (60)$$

Assume first that

$$\frac{3}{2} \frac{x}{\log x} < 3e48^{10} + 1. \quad (61)$$

In this case,

$$\frac{x}{\log x} < \frac{2}{3} (3e48^{10} + 1). \quad (62)$$

Let $C_2 = \frac{2}{3} (3e48^{10} + 1)$. From inequality (62) and lemma 2 in [6], it follows that

$$x < C_2 \log^2 C_2 < \frac{2}{3} (3e48^{10} + 1) \cdot 41^2 < 6 \cdot 10^{20}. \quad (63)$$

In this case, from inequalities (60) and (61), it follows that

$$\frac{y}{\log y} < 2(3e48^{10} + 1). \quad (64)$$
Let $C_3 = 2(3e48^{10} + 1)$. It follows, by inequality (64) and lemma 2 in [6], that
\[ y < C_3 \log^2 C_3 < 2(3e48^{10} + 1) \cdot 42^2 < 1.8 \cdot 10^{21}. \] (65)
Assume now that $y > 2.6 \cdot 10^{21}$. From inequality (59), it follows that $x \geq 2^6$. Moreover, since inequality (65) is a consequence of inequality (61), it follows that
\[ \frac{3}{2} \frac{x}{\log x} \geq 3e48^{10} + 1. \] (66)
From inequalities (60) and (66) it follows that
\[ \frac{3x}{\log x} > \frac{y}{\log y}. \] (67)
We now show that inequality (67) implies $y \leq 1.8 \cdot 10^{21}$. Indeed, assume that $y \geq 4x$. Then inequality (67) implies
\[ \frac{3x}{\log x} > \frac{y}{\log y} \geq \frac{4x}{\log(4x)} = \frac{4x}{\log x + \log 4} \]
or
\[ 3 \log x + 3 \log 4 > 4 \log x \]
or $3 \log 4 > \log x$ which contradicts the fact that $x \geq 2^6$.

Step 2. If $y \geq 3 \cdot 10^{143}$, then $y$ is prime.
Let
\[ y^{x/2} = 2y^{-2}d_1^y - d_2^y \quad \text{or} \quad y^{x/2} = d_1^y - 2^{-y-2}d_2^y. \] (68)
Notice that if $y^{x/2} = 2y^{-2}d_1^y - d_2^y$, then gcd $(2d_1, d_2) = 1$. Let $p \mid y$ be a prime number. Since $p \nmid 2d_1d_2 = x$, it follows, by theorem vdP, that
\[ \frac{x}{2} \leq \max \left( \text{ord}_p (2y^{-2}d_1^y - d_2^y), \text{ord}_p (d_1^y - 2^{-y-2}d_2^y) \right) < 48^{36}e \frac{p}{\log p} \log^2 y \log x. \] (69)
By step 1, it follows that
\[ \frac{1}{4} y < x \leq 2 \cdot 48^{36}e \frac{p}{\log p} \log^2 y \log(4y) < 4 \cdot 48^{36}e \frac{p}{\log p} \log^3 y. \] (70)
Hence,
\[ \frac{y}{\log^3 y} < 16 \cdot 48^{36}e \frac{p}{\log p} < 16 \cdot 48^{36}ep. \] (71)
Suppose that $y$ is not prime. Let $p \mid y$ be a prime such that $p \leq \sqrt{y}$. From inequality (71) it follows that
\[ \frac{\sqrt{y}}{\log^3 y} < 16 \cdot 48^{36}e \]
or
\[
\frac{\sqrt{y}}{\log^3(\sqrt{y})} < 128 \cdot 48^{36}e. \tag{72}
\]
Let \( k = \sqrt{y} \) and \( C_4 = 128 \cdot 48^{36}e. \) By inequality (72) and lemma 2 in [6], it follows that
\[
\sqrt{y} = k < C_4 \log^4 C_4 = 128 \cdot 48^{36}e \cdot 146^4 < 5.3 \cdot 10^{71} \tag{73}
\]
or
\[
y < (5.3 \cdot 10^{71})^2 < 3 \cdot 10^{143} \tag{74}
\]
This last inequality contradicts the assumption that \( y \geq 3 \cdot 10^{143} \).

**Step 3. If** \( y \geq 3 \cdot 10^{143} \), **then** \( x > y \).

Let \( y = p \) be a prime. If \( y^{x/2} = 2^{y-2}d_1^y - d_2^y \), it follows, by Fermat's little theorem that
\[
2^{-1}d_1 - d_2 \equiv 2^{y-2}d_1^y - d_2^y \equiv y^{x/2} \equiv 0 \pmod{p},
\]
therefore
\[
d_1 \equiv 2d_2 \pmod{p}. \tag{75}
\]
On the other hand, if \( y^{x/2} = d_1^y - 2^{y-2}d_2^y \), then
\[
d_1 - 2^{-1}d_2 \equiv d_1^y - 2^{y-2}d_2^y \equiv y^{x/2} \equiv 0 \pmod{p},
\]
therefore
\[
d_2 \equiv 2d_1 \pmod{p}. \tag{76}
\]
Suppose that \( x < y \). From congruences (75) and (76), we conclude that, in both cases, \( x \) is a perfect square. Hence,
\[
y^x = z^2 - (\sqrt{x})^{2y} = \left(z + (\sqrt{x})^y\right) \cdot \left(z - (\sqrt{x})^y\right). \tag{77}
\]
From equation (77) it follows that
\[
\begin{cases}
z - (\sqrt{x})^y = 1 \\
z + (\sqrt{x})^y = y^x
\end{cases} \tag{78}
\]
Hence,
\[
2(\sqrt{x})^y = y^x - 1. \tag{79}
\]
It follows, by equation (79) and theorem BW, that
\[
0 = \log \left| y^x - 2(\sqrt{x})^y \right| = \log(y^x) + \log \left| 1 - 2y^{-x}(\sqrt{x})^y \right| > x \log y - \log 2 - 64^{12}e \log^2 y \log x. \tag{80}
\]
From inequality (80) and Step 1 it follows that
\[ \log 2 + 64^{12} e \log^3 y > x \log y > \frac{y \log y}{4} \]
or
\[ 4 \log 2 + 4 \cdot 64^{12} e \log^3 y > y \log y \]
or
\[ (4 \cdot 64^{12} e + 1) \log^2 y > y \]
or
\[ 4 \cdot 64^{12} e + 1 > \frac{y}{\log^2 y} \] (81)

Let \( C_s = 4 \cdot 64^{12} + 1 \). By inequality (81) and lemma 2 in [6] it follows that
\[ y < C_s \log^3 C_s < (4 \cdot 64^{12} e + 1) \cdot 53^3 < 8 \cdot 10^{27} \] (82)

The last inequality contradicts the fact that \( y \geq 3 \cdot 10^{143} \).

Step 4. Suppose that \( y \geq 3 \cdot 10^{143} \). Let \( y = p \) be a prime. Then, with the notations of Step 1, every solution of equation (4) is of one of the following forms:

1. \( y^{x/2} = 2^{v-2} d_1^y - d_2^y \) with \( y = p, d_1 = 2 + p, d_2 = 1, x = 4 + 2p \)
2. \( y^{x/2} = 2^{v-2} d_1^y - 2^{v-2} d_2^y \) with \( y = p, d_1 = \frac{3p - 1}{2}, d_2 = 1, x = 3p - 1 \)
3. \( y^{x/2} = 2^{v-2} d_1^y - 2^{v-2} d_2^y \) with \( y = p, d_1 = \frac{p - 1}{2}, d_2 = 3, x = 3p - 9 \)

We assume that \( y \geq 3 \cdot 10^{143} \). In this case, \( y = p \) is prime, and \( x > y \). From Step 1 we conclude that \( x < 3y \). Moreover, from the arguments used at Step 1 it follows that \( d_1 > \frac{\sqrt{x}}{2} \). Since \( x = 2d_1d_2 \), it follows that
\[ d_2 < \sqrt{x} < \sqrt{3y} = \sqrt{3p} \]

By the arguments used at Step 3 we may assume that \( x \) is not a perfect square. We distinguish the following cases.

CASE 1. \( d_2 = 1 \). By congruences (75) and (76) it follows that \( d_1 \equiv 2 \pmod{p} \), or \( 2d_1 \equiv 1 \pmod{p} \).

Assume that \( d_1 \equiv 2 \pmod{p} \). Since \( x = 2d_1 \), and \( p = y < x < 3y = 3p \), it follows that \( d_1 = 2 + p \) and \( x = 2d_1 = 4 + 2p \).

Assume that \( 2d_1 \equiv 1 \pmod{p} \). Again, since \( x = 2d_1 \), and \( p = y < x < 3y = 3p \), it follows that \( d_1 = \frac{3p - 1}{2} \), and \( x = 3p - 1 \).

CASE 2. \( d_2 = 2 \). By congruences (75) and (76) it follows that \( d_1 \equiv 4 \pmod{p} \), or \( d_1 \equiv 1 \pmod{p} \). One can easily check that there is no solution in this case. Indeed, if \( d_1 \equiv 4 \pmod{p} \), it follows that \( d_1 \geq p + 4 \). Hence, \( x = 2d_1d_2 \geq 4(p + 4) > 3p = 3y \) which contradicts the fact that \( x < 3y \).
Similar arguments can be used to show that there is no solution for which $d_2 = 2$ and $d_1 \equiv 1 \pmod{p}$.

CASE 3. $d_2 = 3$. By congruences (75) and (76) it follows that $d_1 \equiv 6 \pmod{p}$, or $2d_1 \equiv 3 \pmod{p}$. One can easily check that there is no solution for which $d_1 \equiv 6 \pmod{p}$. Suppose that $2d_1 \equiv 3 \pmod{p}$. Since $p = y < x < 3y = 3p$ and $x = 2d_1d_2 = 6d_1$, it follows easily that $d_1 = \frac{p-3}{2}$, and $x = 3p - 9$.

CASE 4. $d_2 = k \geq 4$.

If $k$ is even, then, by congruences (75) and (76), it follows that $d_1 \equiv 2k \pmod{p}$, or $d_1 \equiv k/2 \pmod{p}$. Since $x$ is not a perfect square it follows that $d_1 \geq p+k/2$, therefore $x \geq 2pk+k^2 > pk \geq 4p > 3p = 3y$ contradicting the fact that $x < 3y$.

If $k$ is odd, then, by congruences (75) and (76), it follows that $d_1 \equiv 2k \pmod{p}$, or $2d_1 \equiv k \pmod{p}$. We conclude that $d_1 \geq \frac{p-k}{2}$, therefore $x = 2d_1d_2 \geq k(p-k)$. Since $k(p-k) > 3p$ for $5 \leq k \leq \sqrt{3p}$ and $p \geq 3 \cdot 10^{143}$, we conclude that $x > 3p = 3y$ contradicting again the fact that $x < 3y$.

Step 5. There are no solutions of equation (2) with $y \geq 3 \cdot 10^{142}$ and $x$ even.

According to Step 4 we need to treat the following cases.

CASE 1.

$$y^{x/2} = 2y^{-2}d_1^y - d_2^y \quad \text{with} \quad y = p, \ d_1 = 2 + p, \ d_2 = 1, \ x = 4 + 2p. \quad (83)$$

Hence,

$$p^{2+p} = 2^{p-2}(2 + p)^p - 1 > 2^{p-3}(2 + p)^p. \quad (84)$$

Taking logarithms in inequality (84) we obtain

$$(2 + p) \log p > (p - 3) \log 2 + p \log(p + 2)$$

or

$$2 \log p + p(\log p - \log(p + 2)) > (p - 3) \log 2. \quad (85)$$

It follows, by inequality (85), that

$$2 \log p > (p - 3) \log 2$$

or

$$p \log 2 < 2 \log p + 3 \log 2 < 5 \log p. \quad (86)$$

Inequality (86) is certainly false for $p = y \geq 3 \cdot 10^{143}$.

CASE 2.

$$y^{x/2} = d_1^y - 2^{y-2}d_2^y \quad \text{with} \quad y = p, \ d_1 = \frac{3p-1}{2}, \ d_2 = 1, \ x = 3p - 1.$$
Hence, 
\[ p^{(3p-1)/2} = \left(\frac{3p-1}{2}\right)^p - 2^{p-2} < \left(\frac{3p-1}{2}\right)^p < \left(\frac{3p}{2}\right)^p \]
or
\[ p^{(p-1)/2} < \left(\frac{3}{2}\right)^p. \tag{87} \]
Taking logarithms in inequality (87) it follows that
\[ \frac{p-1}{2} \log p < p \log 1.5 \]
or
\[ \log p < \frac{2p}{p-1} \log 1.5 < 3 \log 1.5 < \log 1.5^3. \]
It follows that \( p < 1.5^3 < 4 \) which contradicts the fact that \( p \geq 3 \cdot 10^{143} \).

CASE 3.
\[ y^{x/2} = d_1^y - 2^{y-2} d_2^y \quad \text{with} \quad y = p, \quad d_1 = \frac{p-1}{2}, \quad d_2 = 3, \quad x = 3p - 9. \]
Hence,
\[ p^{(3p-9)/2} = \left(\frac{p-3}{2}\right)^p - 2^{p-2} 3^p < \left(\frac{p-3}{2}\right)^p < p^p. \tag{88} \]
From inequality (88) it follows that \( \frac{3p-9}{2} < p \) or \( p < 9 \) which contradicts the fact that \( p = y \geq 3 \cdot 10^{143} \).

The Proof of Theorem 4. The given equation has no solution \((y, z, n)\) with \( n > 1 \) and \( y \) odd, \( y < 5 \). Assume now that \( y \geq 5 \). We may assume that \( n \) is prime. We first show that \( n \) is odd. Indeed, assume that \((y, z)\) is a positive solution of \( y^2 + 2^y = z^2 \) with both \( y \) and \( z \) odd. Then \((z + y)(z - y) = 2^y \). Since \( \gcd(z + y, z - y) = 2 \) it follows that \( z - y = 2 \) and \( z + y = 2^{y-1} \). Hence, \( y = 2^{y-2} - 1 \). However, one can easily check that \( 2^{y-2} - 1 > y \) for \( y \geq 5 \).
Assume now that \( n = p \geq 3 \) is an odd prime. Write
\[ \left(y + 2^{(y-1)/2} \cdot i \sqrt{2} \right) \cdot \left(y - 2^{(y-1)/2} \cdot i \sqrt{2} \right) = z^n \]
Since \( \mathbb{Z}[i \sqrt{2}] \) is euclidian and
\[ \gcd\left(y + 2^{(y-1)/2} \cdot i \sqrt{2}, \quad y - 2^{(y-1)/2} \cdot i \sqrt{2}\right) = 1 \]
it follows that there exists \( a, \quad b \in \mathbb{Z} \) such that
\[ \begin{cases} y + 2^{(y-1)/2} \cdot i \sqrt{2} = (a + bi \sqrt{2})^n \\ y - 2^{(y-1)/2} \cdot i \sqrt{2} = (a - bi \sqrt{2})^n \end{cases} \tag{89} \]
From equations (89) it follows that
\[
y = \frac{(a + bi\sqrt{2})^n + (a - bi\sqrt{2})^n}{2}
\]  (90)
and
\[
2^{(y-1)/2} = \frac{(a + bi\sqrt{2})^n - (a - bi\sqrt{2})^n}{2\sqrt{2}i}
\]  (91)

From equation (90) we conclude that \(a\) is odd. From equation (91), it follows that
\[
2^{(y-1)/2} = b(na^{n-1} + s),
\]
where \(s\) is even. Since both \(n\) and \(a\) are odd, it follows that \(na^{n-1} + s\) is odd as well. Hence, \(b = 2^{(y-1)/2}\). Equation (5) can now be rewritten as
\[
y^2 + 2^y = z^n = \left((a + bi\sqrt{2}) \cdot (a - bi\sqrt{2})\right)^n = (a^2 + 2b^2)^n
\]
or
\[
y^2 + 2^y = (a^2 + 2^y)^n > 2^{ny} \geq 2^{3^y}
\]  (92)

Inequality (92) implies that
\[
y^2 > 2^{3^y} - 2^y = 2^y(2^{2^y} - 1) > 2^y,
\]
which is false for \(y \geq 5\).

Bibliography

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