PROOF OF FUNCTIONAL SMARANDACHE ITERATIONS

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ABSTRACT: The paper makes use of method of Mathematics Analytic to prove Functional Smarandache Iterations of three kinds.

1. Proving Functional Smarandache Iterations of First Kind.

Let \( f: A \rightarrow A \) be a function, such that \( f(x) \leq x \) for all \( x \), and \( \min \{ f(x), x \in A \} \geq m_0 \), different from negative infinity.

Let \( f \) have \( p \geq 1 \) fix points: \( m_0 \leq x_1 < x_2 < \cdots < x_p \). [The point \( x \) is called fix, if \( f(x) = x \).]

Then:

\[
SII(x) = \text{the smallest number of iterations } k \text{ such that } \underbrace{f(f(\cdots f(x)\cdots))}_{\text{iterated } k \text{ times}} = \text{constant.}
\]

Proof: I. When \( A \subseteq Q \) or \( A \subseteq R \), conclusion is false.

Counterexample:
Let \( A = [0, 1] \) with \( f(x) = x^2 \), then \( f(x) \leq x \), and \( x_1 = 0 \), \( x_2 = 1 \) are fix points.

Denote: \( A_n(x) = \underbrace{f(f(\cdots f(x)\cdots))}_{n \text{ times}}, A_1(x) = f(x), (n = 1, 2, \cdots) \).

then \( A_n(x) = x^{2^n} \) ( \( n = 1, 2, \cdots \)).

For any fixed \( x \neq 0 \), \( x \neq 1 \), assumed that the smallest positive integer \( k \) exist, such that \( A_n(x) = a \) (constant), hence, \( A_{k+1}(x) = f(A_k(x)) = f(a) = a \), that is to say \( a \) be fix point.

So \( x^{2^{k+1}} = 0 \) or \( 1 \), \( \Rightarrow x = 0 \) or \( 1 \), this appear contradiction. If \( A \subseteq Z \), let \( A \) be set of all rational number on \( [0, 1] \) with \( f(x) = x^2 \), using the same methods we can also deduce contradictory result.

This shows the conclusion is false where \( A \subseteq Q \) or \( A \subseteq R \).

II. When \( A \subseteq Z \), the conclusion is true.

(1) If \( x = x_i \) (\( x_i \) is fix point, \( i = 1, \cdots p \)). Then \( f(x) = f(x_i) = x_i = A_i(x) \).

So for any positive integer \( n \), \( A_n(x) = x_i \) (\( i = 1, \cdots p \)), \( \Rightarrow SII(x) = 1 \).

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(2). Let \( x \neq x_i \) (\( x \) is fixed, \( i = 1, \cdots, p \)), if \( f(x) = x_i \) (\( i = 1, \cdots, p \)), then \( SI(x) = 1 \), if \( f(x) \neq x_i \) but \( f(f(x)) = A_2(x) = x_i \) (\( i = 1, \cdots, p \)), then \( SI(x) = 2 \). In general, for fixed positive integer \( k \), if \( A_1(x) \neq x_i \), \( A_2(x) \neq x_i \) \( \cdots \) \( A_k(x) \neq x_i \), but \( A_k(x) = x_i \) then \( SI(x) = k \).

(3). Let \( x \neq x_i \) (\( x \) is fixed), and for \( \forall n \in \mathbb{N} \) \( A_n(x) \neq x_i \) (\( i = 1, \cdots, p \)), this case is no exist.

Because \( x \) is fix point, \( m_0 < \cdots < A_n(x) < \cdots < A_1(x) < x \). So sequence \( \{ A_n(x) \} \) is descending and exist boundary, this makes know that \( \{ A_n(x) \} \) is convergent. But, each item of \( \{ A_n(x) \} \) is integer, it is not convergent, this appear contradiction. This shows that the case is no exist.

(4). Let \( x \neq x_i \) (\( x \) is fixed, \( i = 1, \cdots, p \)), if exist the smallest positive integer \( k \) such that \( A_k(x) = a \) (\( a \neq x_i \)), it is yet unable. Because \( A_{k+1}(x) = A_k(x) = a \), \( A_{k+1}(x) = f(A_k(x)) = f(a) = a \), this shows that \( a \) is fix point, namely, \( a = x_i \), this also appear contradiction.

Combining (1), (2), (3) and (4) we have

\[ SI(x) = \text{the smallest number of iterations } k \text{ such that } f^{(k)}(x) = x_i \] (\( x_i \) is fix point, \( i = 1, \cdots, p \)).

This proves Kind 1.

We easily give a simple deduction.

Let \( f: A \to A \) be a function, such that \( f(x) \leq x \) for all \( x \), and \( \min \{ f(x), x \in A \} \geq m_0 \), different from negative infinity.

Let \( f(m_0) = m_0 \), namely, \( m_0 \) is fix point, and only one.

Then: \( SI(x) = \text{the smallest number of iterations } k \text{ such that } f^{(k)}(x) = m_0 \).

2. Proving Functional Smarandache Iterations of Second Kind.

Kind 2.

Let \( g: A \to A \) be a function, such that \( g(x) > x \) for all \( x \), and let \( b > x \).

Then:

\[ SI_2(x, b) = \text{the smallest number of iterations } k \text{ such that } g^{(k)}(x) \geq b. \]
Proof: Firstly, denote: \( B_n(x) = g(g(\cdots g(x)\cdots)), \quad (n=1,2,\ldots) \).

I. Let \( A \subseteq \mathbb{Z}, \) for \( \forall x < b, \ x \in \mathbb{Z}, \) assumed that there are not the smallest positive integer \( k \) such that \( B_k(x) \geq b, \) then for \( \forall n \in \mathbb{N} \) have \( B_n(x) < b, \) so
\[
x < B_1(x) < B_2(x) < \cdots < B_n(x) < \cdots < b.
\]
This makes know that \( \{ B_n(x) \} \) is convergent, but it is not convergent. This appear contradiction, then, there are the smallest \( k \) such that \( B_n(x) \geq b. \)

II. Let \( A \subseteq \mathbb{Q} \) or \( A \subseteq \mathbb{R}. \)

(1). For fixed \( x < b. \) If \( g(x) \geq g(b) > b, \) then \( B_n(x) \geq g(x) > b \quad (n \in \mathbb{N}), \) \( SI2(x,b)=1, \)
\[
\text{if } g(x) < g(b) \text{ but } B_2(x) \geq g(b) > b , \text{ then } B_n(x) \geq g(b) > b \quad (n \geq 2), \quad SI2(x,b) = 2. \text{ In general, if } B_1(x) < g(b), \ B_2(x) < g(b), \ldots \ B_k(x) < g(b), \text{ but } B_k(x) \geq g(b) > b, \text{ then } SI2(x,b)=k.\]

(2). For fixed \( x < b, \) \( B_n(x) < g(b), \quad (n \in \mathbb{N}) \) then
\[
x < B_1(x) < B_2(x) < \cdots < B_n(x) < \cdots < g(b),
\]
so \( \{ B_n(x) \} \) is convergent. Let \( \lim_{n \to \infty} B_n(x) = b^* \quad :\quad B_n(x) < g(b) \quad (n \in \mathbb{N}), \quad :\quad b^* \leq g(b). \)

1). \( b^* = g(b). \quad :\quad \lim_{n \to \infty} B_n(x) = b^*. \quad \) for \( \varepsilon = g(b) - b > 0, \exists \text{ positive integer } k, \) when \( n > k \) such that \( |B_n(x) - g(b)| < \varepsilon. \) So \( B_n(x) > g(b) - \varepsilon = g(b) (g(b) - b) = b. \) That is to say there are the smallest \( k \) such that \( B_n(x) > b. \)
2). \( b^* < g(b). \quad :\quad g(b^*) > b^*, \quad :\quad \{B_n(x)\} \text{ does not converge at } g(b^*). \) So \( \exists \varepsilon_0 > 0, \forall N, \exists n_1, \text{ when } n_1 > N, \text{ such that } |B_{n_1}(x) - g(b^*)| \geq \varepsilon_0, \) then,
\[
B_{n_1}(x) \geq g(b^*) + \varepsilon_0 :\quad B_{n_1}(x) > b^* + \varepsilon_0. \quad \text{On the other hand, } B_n(x) \leq b^* \quad (n \in \mathbb{N}), \quad :\quad B_{n_1}(x) \leq b^* \text{ then } b^* + \varepsilon_0 < B_{n_1}(x) \leq b^*, \text{ but this is unable. This makes know that there is not the case.}
\]
By (1) and (2) we can deduce the conclusion is true in the case of \( A \) belong to \( \mathbb{Q} \) or \( \mathbb{R}. \)

Combining I. and II., we have: for any fixed \( x > b \) there is
\[
SI2(x,b) = \text{the smallest number of iterations } k \text{ such that}
\]
\[
g(g(\cdots g(x)\cdots)) \geq b.
\]
This proves Kind 2.


Kind 3.

Let \( h: A \to A \) be a function, such that \( h(x) < x \) for all \( x, \) and let \( b < x. \)

Then:
$ST3(x,b) = \text{the smallest number of iterations } k \text{ such that }$

\[ h(h(\ldots h(x)\ldots)) \leq b. \]

Using similar methods of proving Kind 2 we also can prove Kind 3, we well not prove again in the place.

We complete the proofs of Functional Smarandache Iterations of all kinds in the place.

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