PROPERTIES OF SMARANDACHE STAR TRIANGLE

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ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION, as follows: Let \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r \) be a set of \( r \) natural numbers and \( p_1, p_2, p_3, \ldots, p_r \) be arbitrarily chosen distinct primes then \( F(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r) \) called the Smarandache Factor Partition of \( (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r) \) is defined as the number of ways in which the number

\[
N = p_1 \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r
\]

could be expressed as the product of its' divisors. For simplicity, we denote \( F(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r) = F(N) \), where

\[
N = p_1 \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r
\]

and \( p_r \) is the \( r^{th} \) prime. \( p_1 = 2, p_2 = 3 \) etc.

Also for the case

\[
\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1
\]

Let us denote

\[
F(1, 1, 1, 1, 1, \ldots) = F(1^n)
\]

In [2] we define the Generalized Smarandache Star Function as follows:
Smarandache Star Function

(1) \[ F'(N) = \sum_{d|N} F'(d) \quad \text{where } d_r | N \]

(2) \[ F''(N) = \sum_{d_r|N} F''(d_r) \]

d_r ranges over all the divisors of N.

If N is a square free number with n prime factors, let us denote

\[ F''(N) = F'(1\#n) \]

Smarandache Generalised Star Function

(3) \[ F^{(n)}(N) = \sum_{d_r|N} F^{(n-1)}(d_r) \quad n > 1 \]

and d_r ranges over all the divisors of N.

For simplicity we denote

\[ F'(Np_1p_2\ldots p_n) = F'(N@1\#n) \quad \text{where} \]

\[ (N,p_i) = 1 \text{ for } i = 1 \text{ to } n \text{ and each } p_i \text{ is a prime.} \]

F'(N@1\#n) is nothing but the Smarandache factor partition of (a number N multiplied by n primes which are coprime to N).

In [2] I had derived a general result on the Smarandache Generalised Star Function. In the present note we define

SMARANDACHE STAR TRIANGLE' (SST) and derive some properties of SST.

DISCUSSION:
DEFINITION: 'SMARANDACHE STAR TRIANGLE' (SST)
As established in [2]

\[ a_{(n,m)} = \frac{1}{m!} \sum_{k=1}^{m} (-1)^{m-k} \cdot m^k \cdot \binom{m}{k} \cdot k^n \]  

we have \( a_{(n,n)} = a_{(n,1)} = 1 \) and \( a_{(n,m)} = 0 \) for \( m > n \). Now if one arranges these elements as follows

\[ a_{(1,1)} \]
\[ a_{(2,1)} \quad a_{(2,2)} \]
\[ a_{(3,1)} \quad a_{(3,2)} \quad a_{(3,3)} \]
\[ \ldots \]
\[ \ldots \]
\[ a_{(n,1)} \quad a_{(n,2)} \quad \ldots \quad a_{(n,n-1)} \quad a_{(n,n)} \]

we get the following triangle which we call the ‘SMARANDACHE STAR TRIANGLE’ in which \( a_{(r,m)} \) is the \( m^{th} \) element of the \( r^{th} \) row and is given by (A) above. It is to be noted here that the elements are the Stirling numbers of the first kind.

\[
\begin{array}{cccc}
1 & & & \\
1 & 1 & & \\
1 & 3 & 1 & \\
1 & 7 & 6 & 1 \\
1 & 15 & 25 & 10 & 1 \\
& & & & \\
\ldots & & & & \\
\end{array}
\]
Some properties of the SST.

(1) The elements of the first column and the last element of each row is unity.

(2) The elements of the second column are $2^{n-1} - 1$, where $n$ is the row number.

(3) Sum of all the elements of the $n^{th}$ row is the $n^{th}$ Bell.

PROOF:
From theorem (3.1) of Ref. [2] we have

$$F'(N@1\#n) = F'(Np_1p_2. . .p_n) = \sum_{m=0}^{n} a_{(n,m)} F_{m*}(N)$$

if $N = 1$ we get $F_{m*}(1) = F_{(m-1)*}(1) = F_{(m-2)*}(1) = . . . = F'(1) = 1$

hence

$$\Gamma'(p_1p_2. . .p_n) = \sum_{r=0}^{n} a_{(n,m)}$$

(4) The elements of a row can be obtained by the following reduction formula

$$a_{(n+1,m+1)} = a_{(n,m)} + (m+1) \cdot a_{(n+1,m+1)}$$

instead of having to use the formula (4.5).

(5) If $N = p$ in theorem (3.1) Ref. [2] we get $F_{m*}(p) = m + 1$. Hence

$$F'(pp_1p_2. . .p_n) = \sum_{m=1}^{n} a_{(n,m)} F_{m*}(N)$$

or

$$B_{n+1} = \sum_{m=1}^{n} (m+1) \cdot a_{(n,m)}$$

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Elements of second leading diagonal are triangular numbers in their natural order.

If \( p \) is a prime, \( p \) divides all the elements of the \( p^{th} \) row except the \( 1^{st} \) and the last, which are unity. This has been established in the following theorem.

**THEOREM (1.1):**

\[ a_{(p,r)} \equiv 0 \pmod{p} \text{ if } p \text{ is a prime and } 1 < r < p \]

**Proof:**

\[ a_{(p,r)} = \frac{1}{r!} \sum_{k=1}^{m} (-1)^{r-k} \cdot \binom{r}{k} \cdot k^p \]

Also

\[ a_{(p,r)} = \frac{1}{(r-1)!} \sum_{k=0}^{r-1} (-1)^{r-1-k} \cdot r-1 \cdot \binom{r-1}{k} \cdot (k+1)^{p-1} \]

\[ a_{(p,r)} = \frac{1}{(r-1)!} \sum_{k=0}^{r-1} \left[ (-1)^{r-1-k} \cdot r-1 \cdot \binom{r-1}{k} \cdot ((k+1)^{p-1} - 1) \right] + \]

\[ \frac{1}{(r-1)!} \sum_{k=0}^{r-1} (-1)^{r-1-k} \cdot r-1 \cdot \binom{r-1}{k} \]

Applying Fermat's little theorem, we get

\[ a_{(p,r)} = \text{a multiple of } p + 0 \]

\[ \Rightarrow a_{(p,r)} \equiv 0 \pmod{p} \]

**COROLLARY: (1.1)**

\[ F(1\#p) \equiv 2 \pmod{p} \]

\[ a_{(p,1)} = a_{(p,p)} = 1 \]
(8) The coefficient of the $r^{th}$ term $b_{(n,r)}$ in the expansion of $x^n$ as

$$x^n = b_{(n,1)} x + b_{(n,2)} x(x-1) + b_{(n,3)} x(x-1)(x-2) + \cdots + b_{(n,r)} x^r + \cdots + b_{(n,n)} x^n$$

is equal to $a_{(n,r)}$.

**THEOREM (1.2):** $B_{3n+2}$ is even else $B_k$ is odd.

From theorem (2.5) in REF. [1] we have

$$F'(Nq_1q_2) = F'^*(N) + F'^**(N)$$

where $q_1$ and $q_2$ are prime.

and $(N,q_1) = (N,q_2) = 1$

let $N = p_1p_2p_3 \ldots p_n$ then one can write

$$F'\left( p_1p_2p_3 \ldots p_nq_1q_2 \right) = F'^*(p_1p_2p_3 \ldots p_n) + F'^**(p_1p_2p_3 \ldots p_n)$$

or

$$F(1#(n+2)) = F(1#(n+1)) + F**(1#n)$$

but

$$F**(1#n) = \sum_{r=0}^{n} {n \choose r} 2^{n-r} F(1#r)$$

$$F**(1#n) = \sum_{r=0}^{n-1} {n \choose r} 2^{n-r} F(1#r) + F(1#n)$$

the first term is an even number say $= E$, This gives us

$$F(1#(n+2)) - F(1#(n+1)) - F(1#n) = E$$

an even number. \(\text{---(1.1)}\)

Case- I: $F(1#n)$ is even and $F(1#(n+1))$ is also even $\Rightarrow$
F(1#(n+2)) is even.

Case -II: F(1#n) is even and F(1#(n+1)) is odd $\Rightarrow$ F(1#(n+2)) is odd.

again by (1.1) we get

$F(1#(n+3)) - F(1#(n+2)) - F(1#(n+1)) = E$, $\Rightarrow$ F(1#(n+3)) is even. Finally we get

F(1#n) is even $\iff$ F(1#(n+3)) is even.

we know that F(1#2) = 2 $\Rightarrow$ F(1#2), F(1#5), F(1#8),...are even.

$\Rightarrow$ $B_{3n+2}$ is even else $B_k$ is odd

This completes the proof.

REFERENCES:


[3] "The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.