PROPOSED PROBLEM

by J. Thompson

Calculate:

\[
\lim_{n \to \infty} \left( 1 + \sum_{k=1}^{n} \frac{1}{\eta(k)} - \log \eta(n) \right)
\]

where \( \eta(n) \) is Smarandache Function: the smallest integer \( m \) such that \( m! \) is divisible by \( n \).

Solution:

We know that \( \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) \) converges to \( e \) for \( n \to \infty \).

It's easy to show that for \( k \geq 2 \), \( \eta(k) \leq k \). More, for \( k \) a composite number \( \geq 10 \), \( \eta(k) \leq k / 2 \). Also, if \( p > 4 \) then: \( \eta(p) = p \) if and only if \( p \) is prime.

\[
\sum_{k=10}^{n} \sum_{k=1}^{n} \frac{1}{\eta(k)} - \log \eta(n) = \left( \sum_{k=10}^{n} \frac{1}{k} - \log n \right) + \sum_{k \text{ prime}}^{n} \frac{1}{k} \to e + \infty = \infty
\]

because for any prime number \( p \) there exists a composite number \( p-1 \) such that

\[
\frac{1}{p-1} > \frac{1}{p}
\]

thus:

\[
\sum_{k=10}^{n} \frac{1}{k} = \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{18} + \ldots + \frac{1}{n} > \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \ldots + \frac{1}{p(n)} \to \infty
\]

where \( p(n) \) is the greatest prime number less than \( n \).

We took out the first nine terms of that series, the limit of course didn’t change.

Reference:


see Mathematical Review: 82a:03012.

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