SOLUTION OF TWO QUESTIONS CONCERNING THE DIVISOR FUNCTION AND THE PSEUDO-SMARANDACHE FUNCTION

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Abstract In this paper we completely solve two questions concerning the divisor function and the pseudo-Smarandache function.

Key words divisor function, pseudo-Smarandache function, functional equation

1 Introduction

Let \( \mathbb{N} \) be the set of all positive integers. For any \( n \in \mathbb{N} \), let

\[
(1) \quad d(n) = \sum_{d|n} 1,
\]

\[
(2) \quad Z(n) = \min \{ a | a \in \mathbb{N}, n | \sum_{j=1}^{a} j \}
\]

Then \( d(n) \) and \( Z(n) \) are called the divisor function and the pseudo-Smarandache function of \( n \), respectively. In \([1]\), Ashbacher posed the following unsolved questions.

**Question 1** How many solutions \( n \) are there to the functional equation.

\[
(3) \quad Z(n) = d(n), n \in \mathbb{N}?
\]

**Question 2** How many solutions \( n \) are there to the functional equation.

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In this paper we completely solve the above questions as follows.

**Theorem 1** The equation (3) has only the solutions $n = 1, 3$ and $10$.

**Theorem 2** The equation (4) has only the solution $n = 56$.

2 **Proof of Theorem 1**

A computer search showed that (3) has only the solutions $n = 1, 3$ and $10$ with $n \leq 10000$ (see [1]).

We now let $n$ be a solution of (3) with $n \neq 1, 3$ or $10$. Then we have $n > 10000$. Let

$$n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$$

be the factorization of $n$. By [2, Theorem 273], we get from (1) and (5) that

$$d(n) = (r_1 + 1)(r_2 + 1) \cdots (r_k + 1).$$

On the other hand, since $\sum_{j=1}^a j = a(a + 1)/2$ for any $a \in \mathbb{N}$, we see from (2) that $n | Z(n)(Z(n) + 1)/2$. It implies that $Z(n)(Z(n) + 1)/2 \geq n$. So we have

$$Z(n) \geq \sqrt{2n + \frac{1}{4} - \frac{1}{2}}$$

Hence, by (3), (5), (6) and (7), we get

$$1 > \sqrt{2 \prod_{i=1}^k \frac{p_i^{r_i/2}}{r_i + 1}} - \frac{1}{2} \prod_{i=1}^k \frac{1}{r_i + 1}$$

If $p_1 > 3$, then from (8) we get $p_1 \geq 5$ and

$$1 \geq \sqrt{2 \left(\frac{\sqrt{5}}{2}\right)^k - \frac{1}{2^{k+1}}} > 1,$$
a contradiction. Therefore, if (8) holds, then either $p_1 = 2$ or $p_1 = 3$. By
the same method, then $n$ must satisfy one of the following conditions.

(i) $p_1 = 2$ and $r_1 \leq 4$.
(ii) $p_1 = 3$ and $r_1 = 1$.

However, by (8), we can calculate that $n < 10000$, a contradiction. Thus, the theorem is proved.

3 Proof of Theorem 2

A computer search showed that (4) has only the solution $n = 56$ with $n
\leq 10000$ (see \cite{11}). We now let $n$ be a solution of (4) with $n \neq 56$. Then we
have $n > 10000$. We see from (4) that

$$Z(n) \equiv -d(n) \pmod{n}$$

It implies that.

$$Z(n) + 1 \equiv 1 - d(n) \pmod{n}$$

By the proof of Theorem 1, we have $n \mid Z(n)(Z(n) + 1)/2$, by (2). It can
be written as

$$Z(n)(Z(n) + 1) \equiv 0 \pmod{n}.$$ 

Substituting (9) and (10) into (11), we get

$$d(n)(d(n) - 1) \equiv 0 \pmod{n}.$$ 

Notice that $d(n) > 1$ if $n > 1$. We see from (12) that

$$d(n) > 1 \quad \text{and} \quad (d(n))^2 > n$$

Let (5) be the factorization of $n$. By (5), (6) and (13), we obtain

$$1 > \prod_{i=1}^{r_i} \frac{p_i^{r_i}}{(r_i + 1)^2}$$
On the other hand, it is a well known fact that \(Z(p^r) \geq p^r - 1 > (r + 1)^2\) for any prime power \(p^r\) with \(p^r > 32\). We find from (14) that \(k \geq 2\).

If \(p_1 > 3\), then \(p_1/(r_1 + 1)^2 \geq 5/4 > 1\) for \(i = 1, 2, \ldots, k\). It implies that if (14) holds, then either \(p_1 = 2\) or \(p_1 = 3\). By the same method, then \(n\) must satisfy one of the following conditions:

(i) \(p_1 = 2, p_2 = 3\) and \((r_1, r_2) = (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (1, 2), (2, 2), (3, 2), (4, 2)\) or \((5, 2)\).

(ii) \(p_1 = 2, p_2 > 3\) and \(r_1 \leq 5\).

(iii) \(p_1 = 3\) and \(r_1 = 1\).

However, by (14), we can calculate that \(n < 10000\), a contradiction. Thus, the theorem is proved.

References


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