SOME CONJECTURES ON PRIMES (I)

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Abstract. For any complex number \( x \), let \( \exp(x) = e^x \). For any positive integer \( n \), let \( p_n \) be the \( n \)th prime. In this paper we prove that

\[
\exp\left(\sqrt{\frac{n+1}{p_{n+1}}}\right) / \exp\left(\frac{p_n}{n}\right) < \exp\left(\frac{3}{5}\right) / \exp\left(\frac{3}{2}\right)
\]

Key words: prime, inequality.

For any complex number \( x \), let \( \exp(x) = e^x \). For any positive integer \( n \), let \( p_n \) be the \( n \)th prime. Recently, Russo [2] proposed the following conjecture:

Conjecture For any positive integer \( n \),

\[
\exp\left(\sqrt{\frac{n+1}{p_{n+1}}}\right) \exp\left(\frac{3}{5}\right) < \exp\left(\frac{p_n}{n}\right) \exp\left(\frac{3}{2}\right).
\]

In [2], Russo verified (1) for \( p_n \leq 10^7 \). In this paper we completely solve the above-mentioned conjecture as follows.

Theorem For any positive integer \( n \), the inequality (1) holds.

Proof We may assume that \( p_n > 10^7 \). Then we have \( n > 1000 \).

It is a well known fact that

\[
p_n > n \log n,
\]

for any positive integer \( n \) (see [1]). By (2), we get
On the other hand, since $p_{n+1} > n+1$, we get

$$\exp\left(\sqrt{\frac{p_n}{n}}\right) > \exp\left(\sqrt{\log n}\right) > \exp\left(\sqrt{\log 1000}\right) > \exp(2.6).$$

The combination of (3) and (4) yields

$$\exp\left(\frac{3}{2} - \frac{3}{5} + \frac{n+1}{p_n} + 1\right) \leq \exp(1.5).$$

Thus, by (5), we get (1) immediately. The theorem is proved.

References


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