SOME CONJECTURES ON PRIMES (III)

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Abstract. For any positive integer \( n \), let \( p_n \) be the \( n \)th prime. In this paper we prove that the equality

\[
\left( \frac{\sqrt{p_n} - \log p_{n+1}}{\sqrt{p_n} - \log p_n} \right) \geq \left( \frac{\sqrt{3} - \log 5}{\sqrt{5} - \log 3} \right)
\]

for any \( n \).

Key words: prime, inequality.

For any positive integer \( n \), let \( p_n \) be the \( n \)th prime. Recently, Russo \([2]\) proposed the following conjecture:

Conjecture For any positive integer \( n \),

\[
\frac{\sqrt{p_n} - \log p_{n+1}}{\sqrt{p_{n-1}} - \log p_n} \geq \frac{\sqrt{3} - \log 5}{\sqrt{5} - \log 3}.
\]

In \([2]\), Russo verified the equality (1) holds for \( p_n \leq 10^7 \). In this paper, we completely solve the above-mentioned problem as follows.

Theorem For any positive integer \( n \), the equality (1) holds.

Proof We may assume that \( p_n > 10^7 \). Since

\[
\frac{\sqrt{3} - \log 5}{\sqrt{5} - \log 3} < 0.11,
\]

if (1) is false, then from (2) we get

\[
\sqrt{p_n} < \log p_{n+1} + 0.11 \sqrt{p_{n+1}}.
\]

It is a well known fact that
for any positive integer $n$ (see [1, Theorem 245]). Substitute (4) into (3), we obtain

$$0.84\sqrt{p_n} < \log(2p_n)$$

However, (5) is impossible for $p_n > 10^7$. Thus, the theorem is proved.

References


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