In this paper we continue the algebraic consideration begun in [2]. As it was sun, two of the proprieties of Smarandache’s function are hold:

1. \( S \) is a surjective function;
2. \( S([m,n]) = \max \{ S(m), S(n) \} \), where \([m,n]\) is the smallest common multiple of \(m\) and \(n\).

That is on \( \mathbb{N} \) there are considered both of the divisibility order \( \leq_d \) having the known properties and the total order with the usual order \( \leq \) with all its properties. \( \mathbb{N} \) has also the algebraic usual operations \( \cdot \) and \( + \). For instance:

\[
a \leq b \iff (\exists)\ u \in \mathbb{N} \text{ so that } b = a + u.
\]

Here we can stand out:

\( \vdash \) the universal algebra \((\mathbb{N}^*, \Omega)\), the set of operations is \( \Omega = \{ \vee_d, \phi_0 \} \) where \( \vee_d : (\mathbb{N}^*)^2 \rightarrow \mathbb{N}^* \) is given by \( m \vee_d n = [m, n] \), and \( \phi_0 : (\mathbb{N}^*)^0 \rightarrow \mathbb{N}^* \) the null operation that fixes 1-unique particular element with the role of neutral element for \( \vee_d \)-that means \( \phi_0 (\{0\}) = 1 \) and \( 1 = e_{\vee_d} \);

\( \vdash \) the universal algebra \((\mathbb{N}^*, \Omega')\), the set of operations is \( \Omega' = \{ \vee, \psi_0 \} \) where \( \vee : \mathbb{N}^2 \rightarrow \mathbb{N}^* \) is given by \( x \vee y = \sup \{x, y\} \) and \( \psi_0 : \mathbb{N}^0 \rightarrow \mathbb{N}^* \) a null operation with \( \psi_0 (\{0\}) = 0 \) the unique particular element with the role of neutral element for \( \vee \), so \( 0 = e_{\vee} \).

We observe that the universal algebras \((\mathbb{N}^*, \Omega)\) and \((\mathbb{N}^*, \Omega')\) are of the same type:

\[
\begin{pmatrix}
\vee_d & \phi_0 \\
2 & 0
\end{pmatrix}
= \begin{pmatrix}
\vee & \psi_0 \\
2 & 0
\end{pmatrix}
\]

and with the similarity (bijective) \( \vee_d \iff \vee \) and \( \phi_0 \iff \psi_0 \), Smarandache’s function \( S : \mathbb{N}^* \rightarrow \mathbb{N}^* \) is a morphism surjective between them

\[
S (x \vee_d y) = S(x) \vee S(y), \forall x, y \in \mathbb{N}^* \text{ from (2) and}
\]

\[
S (\phi_0 (\{0\})) = \psi_0 (\{0\}) \iff S(1) = 0.
\]
Problem 3. If \( S : \mathbb{N}^* \to \mathbb{N} \) is Smarandache's function defined as we know by

\[
S(n) = m \iff m = \min \{ k : n \text{ divides } k! \}
\]

and \( I \) is a some set, then there exists an unique \( s : (\mathbb{N}^*)^I \to \mathbb{N}^I \) a surjective morphism between the universal algebras \((\mathbb{N}^*)^I, \Omega)\) and \((\mathbb{N}^I, \Omega^I)\) so that \( p_i \circ s = \circ \circ \pi_i \), for \( i \in I \), where \( p_j : \mathbb{N}^I \to \mathbb{N} \) defined by \( a = \{ a_i \}_{i \in I} \in \mathbb{N}^I, p_j (a) = a_j \), for each \( j \in I \), \( p_j \) are the canonical projections, morphismes between \((\mathbb{N}^I, \Omega^I)\) and \((\mathbb{N}, \Omega^I)\)-universal algebras of the same kind and \( \tilde{p}_j : (\mathbb{N}^*)^I \to \mathbb{N}^* \) analogously between \((\mathbb{N}^*)^I, \Omega^I)\) and \((\mathbb{N}^*, \Omega)\). We shall go over the following three steps in order to justify the assumption:

Theorem 0.1. Let by \((\mathbb{N}, \Omega)\) is an universal algebra more complexe with

\[
\Omega = \{ V_d, \wedge_d, \varphi_0, \varphi_0 \}
\]

of the kind \( \tau : \Omega \to \mathbb{N} \) given by

\[
\tau = \left( \begin{array}{cccc}
V_d & \wedge_d & \varphi_0 & \varphi_0 \\
2 & 2 & 0 & 0 \\
\end{array} \right)
\]

where \( V_d \) and \( \varphi_0 \) are defined as above and \( \wedge_d : \mathbb{N}^2 \to \mathbb{N} \), for each \( x, y \in \mathbb{N}, x \wedge_d y = (x, y) \) where \( (x, y) \) is the biggest common divisor of \( x \) and \( y \) and \( \varphi_0 : \mathbb{N}^0 \to \mathbb{N} \) is the null operation that fixes \( 0 \) an unique particular element having the role of the neutral element for \( \wedge_d \) i.e. \( \varphi_0 (\{0\}) = 0 \) so \( 0 = e_{\wedge_d} \) and \( I \) a set. Then \((\mathbb{N}, \Omega)\) with \( \tilde{\Omega} = \{ \omega_1, \omega_2, \omega_0, \tilde{\omega}_0 \} \) becomes an universal algebra of the same kind as \((\mathbb{N}, \Omega)\) and the canonical projections become surjective morphismes between \((\mathbb{N}^I, \tilde{\Omega})\) and \((\mathbb{N}, \Omega)\), an universal algebra that satisfies the following property of universality:

\((U)\): for every \((A, \tilde{\Omega})\) with \( \tilde{\Omega} = \{ \top, \bot, \sigma_0, \sigma_0 \} \) an universal algebra of the same kind

\[
\tau = \left( \begin{array}{cccc}
\top & \bot & \sigma_0 & \sigma_0 \\
2 & 2 & 0 & 0 \\
\end{array} \right)
\]

and \( u_i : A \to \mathbb{N}, \) for each \( i \in I, \) morphismes between \((A, \tilde{\Omega})\) and \((\mathbb{N}, \Omega)\), exists an unique \( u : A \to \mathbb{N}^I \) morphism between the universal algebras \((A, \tilde{\Omega})\) and \((\mathbb{N}^I, \tilde{\Omega})\) so that \( p_j \circ u = u_j, \) for each \( j \in I, \) where \( p_j : \mathbb{N}^I \to \mathbb{N} \) with each \( a = \{ a_i \}_{i \in I} \in \mathbb{N}^I, p_j (a) = a_j \), for each \( j \in I \) are the canonical projections morphismes between \((\mathbb{N}^I, \tilde{\Omega})\) and \((\mathbb{N}, \Omega)\).
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Proof. Indeed \((\mathbb{N}', \bar{\Omega})\) with \(\bar{\Omega} = \{\omega_1, \omega_2, \omega_0, \bar{\omega}_0\}\) becomes an universal algebra because we can well define:

\[\begin{align*}
\omega_1 &: (\mathbb{N}')^2 \to \mathbb{N}^I \text{ by each } a = \{a_i\}_{i \in I}, b = \{b_i\}_{i \in I} \in \mathbb{N}; \omega_1 (a, b) = \{a_i \lor_d b_i\}_{i \in I} \in \mathbb{N}^I \\
\omega_2 &: (\mathbb{N}')^2 \to \mathbb{N}^I \text{ by } \omega_2 (a, b) = \{a_i \land_d b_i\}_{i \in I} \in \mathbb{N}^I \\
\omega_0 &: (\mathbb{N}')^0 \to \mathbb{N}^I \text{ with } \omega_0 (\emptyset) = \{e_i = 1\}_{i \in I} \in \mathbb{N}^I
\end{align*}\]

an unique particular element (the family with all the components equal with 1) fixed by \(\omega_0\) and having the role of neutral for the operation \(\omega_1\) noted with \(e''_I\) and then \(\bar{\omega}_0 : (\mathbb{N}')^0 \to \mathbb{N}^I \) with \(\bar{\omega}_0 (\emptyset) = \{e_i = 0\}_{i \in I}\) an unique particular element fixed by \(\bar{\omega}_0\) but having the role of neutral for the operation \(\omega_2\) noted \(e''_0\) (the verifies are immediate).

The canonical projections \(p_j : \mathbb{N}^I \to \mathbb{N}\), defined as above, become morphisms between \((\mathbb{N}', \bar{\Omega})\) and \((\mathbb{N}, \Omega)\). Indeed the two universal algebras are of the same kind

\[
\left(\begin{array}{ccc}
\omega_1 & \omega_2 & \omega_0 & \bar{\omega}_0 \\
2 & 2 & 0 & 0
\end{array}\right) = \left(\begin{array}{ccc}
\lor_d & \land_d & \varphi_0 & \bar{\varphi}_0 \\
2 & 2 & 0 & 0
\end{array}\right)
\]

and with the similarity (bijective) \(\omega_1 \leftrightarrow \lor_d; \omega_2 \leftrightarrow \land_d; \omega_0 \leftrightarrow \varphi_0; \bar{\omega}_0 \leftrightarrow \bar{\varphi}_0\)

we observe first that for each \(a, b \in \mathbb{N}^I\), \(p_j (\omega_1 (a, b)) = p_j (a) \lor_d p_j (b)\), for each \(j \in I\) because \(a = \{a_i\}_{i \in I}, b = \{b_i\}_{i \in I}\), \(p_j (\omega_1 (a, b)) = p_j (\{a_i \lor_d b_i\}_{i \in I}) = a_j \lor_d b_j\) and \(p_j (a) \lor_d p_j (b) = p_j (\{a_i\}_{i \in I}) \lor_d p_j (\{b_i\}_{i \in I}) = a_j \lor_d b_j\) and then \(p_j (\omega_0 (\emptyset)) = \varphi_0 (\emptyset) \leftrightarrow p_j (\{e_i = 1\}_{i \in I}) = 1 \leftrightarrow p_j (e_i) = e_i \lor_d\) analogously we prove that \(p_j\), for each \(j \in I\) keeps the operations \(\omega_2\) and \(\bar{\omega}_0\), too. So, it was built the universal algebra \((\mathbb{N}', \bar{\Omega})\) with \(\bar{\Omega} = \{\omega_1, \omega_2, \omega_0, \bar{\omega}_0\}\) of the kind \(\tau\) described above.

We prove the property of universality \((U)\).

We observe for this purpose that the \(u_i\) morphisms for each \(i \in I\), preserves the coditions: for each \(x, y \in S, u_i (x \lor y) = u_i (x) \lor_d u_i (y); u_i (x \land y) = u_i (x) \land_d u_i (y); u_i (\sigma_0 (\emptyset)) = \varphi_0 (\emptyset) \leftrightarrow u_i (e_T) = e_i \lor_d = 1\) and \(u_i (\sigma_0 (\emptyset)) = \bar{\varphi}_0 (\emptyset) \leftrightarrow u_i (e_\top) = e_i \land_d = 0\) which show also the similarity (bijective) between \(\bar{\Omega}\) and \(\Omega\). We also observe that \((S, \bar{\Omega})\) and \((\mathbb{N}', \bar{\Omega})\) are of the same kind and there is a similarity (bijective) between \(\bar{\Omega}\) and \(\bar{\Omega}\) given by \(T \leftrightarrow \omega_1; \bot \leftrightarrow \omega_2; \sigma_0 \leftrightarrow \omega_0; \sigma_0 \leftrightarrow \bar{\omega}_0\).

We define the corespondance \(u : A \to \mathbb{N}'\) by \(u(x) = \{u_i (x)\}_{i \in I}\).

\(u\) is the function:

- for each \(x \in A, (\exists) u_i (x) \in \mathbb{N}\) for each \(i \in I\) \((u_i\)-functions) so \((\exists) \{u_i (x)\}_{i \in I}\) that can be imagines for \(x\);
\[ x_1 = x_2 \implies u(x_1) = u(x_2) \text{ because } x_1 = x_2 \text{ and } u_i\text{-functions lead to } u_i(x_1) = u_i(x_2) \text{ for each } i \in I \implies \{u_i(x_1)\}_{i \in I} = \{u_i(x_2)\}_{i \in I} \implies u(x_1) = u(x_2). \]

\( u \) is a morphism: for each \( x, y \in A \), \( u(x \wedge y) = \{u_i(x \wedge y)\}_{i \in I} = \{u_i(x) \vee_d u_i(y)\}_{i \in I} = \omega_1(\{u_i(x)\}_{i \in I}, \{u_i(y)\}_{i \in I}) = \omega_1(u(x), u(y)). \) Then \( u(\sigma_0(\emptyset)) = \sigma_0(\emptyset) \implies u(e_\tau) = e_u \) because for each \( \{a_i\}_{i \in I} \in \mathbb{N}^I, \omega_1(\{a_i\}_{i \in I}, \{u_i(e_\tau)\}_{i \in I}) = \{a_i \vee_d u_i(e_\tau)\}_{i \in I} = \{a_i\}_{i \in I}. \)

Analogously we prove that \( u \) keeps the operations: \( \land \) and \( \lor \).

Besides the condition \( p_j \circ u = u_j \), for each \( j \in I \) is verified (by the definition: for each \( x \in S \), \( (p_j \circ u)(x) = p_j(u(x)) = p_j(\{u_i(x)\}_{i \in I}) = u_j(x) \)).

For the singleness of \( u \) we consider \( u \) and \( \overline{u} \), two morphisms so that \( p_j \circ u = u_j \) (1) and \( p_j \circ \overline{u} = u_j \) (2), for every \( j \in I \). Then for every \( x \in A \), if \( u(x) = \{u_i(x)\}_{i \in I} \) and \( \overline{u}(x) = \{z_i\}_{i \in I} \) we can see that \( y_j = u_j(x) = (p_j \circ \overline{u})(x) = p_j(\{z_i\}_{i \in I}) = z_j \), for every \( j \in I \) i.e. \( u(x) = \overline{u}(x) \), for every \( x \in A \iff u = \overline{u} \).

Consequence. Particularly, taking \( A = \mathbb{N}^I \) and \( u_i = p_i \) we obtain: the morphism \( u : \mathbb{N}^I \rightarrow \mathbb{N}^I \) verifies the condition \( p_j \circ u = p_j \), for every \( j \in I \), if and only if, \( u = 1_{\mathbb{N}^I} \).

The property of universality establishes the universal algebra \( (\mathbb{N}^I, \overline{\Omega}) \) until an isomorphism as it results from:

**Theorem 0.2.** If \( (P, \Omega) \) is an universal algebra of the same kind as \( (\mathbb{N}, \Omega) \) and \( p_i : P \rightarrow \mathbb{N}, i \in I \) a family of morphisms between \((P, \Omega)\) and \((\mathbb{N}, \Omega)\) so that for every universal algebra \( (A, \overline{\Omega}) \) and every morphism \( u_i : A \rightarrow \mathbb{N}, \) for every \( i \in I \) between \((A, \overline{\Omega})\) and \((\mathbb{N}, \Omega)\) it exists an unique morphisme \( u : A \rightarrow P \) with \( p_j \circ u = u_j \), for every \( i \in I \), then it exists an unique isomorphism \( f : P \rightarrow \mathbb{N}^I \) with \( p_i \circ f = p_i, \) for every \( i \in I \).

Proof. From the property of universality of \( (\mathbb{N}^I, \overline{\Omega}) \) it results an unique \( f : P \rightarrow \mathbb{N}^I \) so that for every \( i \in I, p_i \circ f = p_i \) with \( f \) morphisme between \((P, \Omega)\) and \((\mathbb{N}^I, \overline{\Omega})\). Applying now the same property of universality to \((P, \Omega) \implies exists an unique \( \overline{f} : \mathbb{N}^I \rightarrow P \) so that \( p_i \circ \overline{f} = p_i, \) for every \( i \in I \) with \( \overline{f} \) morphisme between \((\mathbb{N}^I, \overline{\Omega})\) and \((P, \Omega)\). Then \( p_i \circ \overline{f} = p_i \iff p_i \circ (f \circ \overline{f}) = p_i, \) using the last consequence, we get \( f \circ \overline{f} = 1_{\mathbb{N}^I}. \) Analogously, we prove that \( f \circ \overline{f} = 1_P \) from where \( \overline{f} = f^{-1} \) and the morphisme \( f \) becomes isomorphism.

We could emphasize other properties (a family of finite support or the case \( I^-\text{filter} \)) but we remain at these which are strictly necessary to prove the proposed assertion (Problem 3).

b) Firstly it was built \( (\mathbb{N}^I, \overline{\Omega}) \) being an universal algebra more complexe (with four operations). We try now a similar construction starting from \((\mathbb{N}, \Omega^*) \) with \( \Omega^* =
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\( (\lor, \land, \psi_0) \) with "\( \lor \)" and "\( \psi_0 \)" defined as above and \( \land : \mathbb{R}^2 \to \mathbb{R} \) with \( x \land y = \inf \{x, y\} \) for every \( x, y \in \mathbb{R} \). \( \blacksquare \)

**Theorem 0.3.** Let by \((\mathbb{R}, \Omega^*)\) the above universal algebra and \( I \) a set. Then:

(i) \((\mathbb{R}^I, \theta)\) with \( \theta = \{\theta_1, \theta_2, \theta_0\} \) becomes an universal algebra of the same kind \( \tau \) as \((\mathbb{R}, \Omega^*)\) so \( \tau : \theta \to \mathbb{R} \) is

\[
\tau = \left( \begin{array}{ccc}
\theta_1 & \theta_2 & \theta_0 \\
2 & 2 & 0
\end{array} \right);
\]

(ii) For every \( j \in I \) the canonical projection \( p_j : \mathbb{R}^I \to \mathbb{R} \) defined by every \( a = \{a_i\}_{i \in I} \in \mathbb{R}^I \), \( p_j(a) = a_j \) is a surjective morphisme between \((\mathbb{R}^I, \theta)\) and \((\mathbb{R}, \Omega^*)\) and \( \ker p_j = \{a \in \mathbb{R}^I : a = \{a_i\}_{i \in I} \) and \( a_j = 0\} \) where by definition we have \( \ker p_j = \{a \in \mathbb{R}^I : p_j(a) = e_\nu\} \);

(iii) For every \( j \in I \) the canonical injection \( q_j : \mathbb{R} \to \mathbb{R}^I \) for every \( x \in \mathbb{R} \), \( q_j(x) = \{a_i\}_{i \in I} \) where \( a_i = 0 \) if \( i \neq j \) and \( a_j = x \) is an injective morphisme between \((\mathbb{R}, \Omega^*)\) and \((\mathbb{R}^I, \theta)\) and \( q_j(\mathbb{R}) = \{\{a_i\}_{i \in I} : a_i = 0, \forall i \in I - \{j\}\} \);

(iv) If \( j, k \in I \) then:

\[
p_j \circ q_k = \begin{cases}
\text{O-the null morphisme} & \text{for } j \neq k, \\
1_{\mathbb{R}}-\text{the identical morphisme} & \text{for } j = k.
\end{cases}
\]

**Proof.** (i) We well define the operations \( \theta_1 : (\mathbb{R}^I)^2 \to \mathbb{R}^I \) by \( \forall a = \{a_i\}_{i \in I} \in \mathbb{R}^I \) and \( b = \{b_i\}_{i \in I} \in \mathbb{R}^I \), \( \theta_1(a, b) = \{a_i \lor b_i\}_{i \in I} \); \( \theta_2 : (\mathbb{R}^I)^2 \to \mathbb{R}^I \) by \( \theta_2(a, b) = \{a_i \land b_i\}_{i \in I} \) and \( \theta_0 : (\mathbb{R}^I)^0 \to \mathbb{R}^I \) by \( \theta_0(\{\emptyset\}) = \{e_i = 0\}_{i \in I} \) an unique particular element fixed by \( \theta_0 \), but with the role of neutral element for \( \theta_1 \) and noted \( e_\theta \) (the verifications are immediate).

(ii) The canonical projections are proved to be morphismes (see the step a)), they keep all the operations and

\[
\ker p_j = \{a = \{a_i\}_{i \in I} \in \mathbb{R}^I : p_j(a) = e_\nu\} = \{a \in \mathbb{R}^I : a_j = 0\}.
\]

(iii) For every \( x, y \in \mathbb{R} \), \( q_j(x \lor y) = \{c_i\}_{i \in I} \) where \( c_i = 0 \) for every \( i \neq j \) and \( c_j = x \lor y \) and

\[
\theta_1 \left( \left\{ \begin{array}{l}
\{a_i = 0, \quad \forall i \neq j\} \\
\{a_j = x\}
\end{array} \right\} \right) = \left\{ \begin{array}{l}
\{c_i = 0, \quad \forall i \neq j\} \\
\{c_j = x \lor y\}
\end{array} \right\}
\]

i.e. \( q_j(x \lor y) = \theta_1(q_j(x), q_j(y)) \) with \( j \in I \), therefore \( q_j \) keeps the operation "\( \lor \)" for every \( j \in I \). Then \( q_j(\psi(\emptyset)) = \theta_0(\{\emptyset\}) \iff q_j(e_\nu) = \{e_i = 0\}_{i \in I} \iff q_j(0) = \{e_i = 0\}_{i \in I} = e_\theta \), because \( \forall a = \{a_i\}_{i \in I} \in \mathbb{R}^I \), \( \theta_1(q_j(0), a) = \theta_1(\{e_i = 0\}_{i \in I}, \{a_i\}_{i \in I}) = e_\theta \)
For every }x \in \mathbb{N}, \ (p_j \circ q_k)\(x) = p_j\(q_k\(x)) = p_j\left(\begin{array}{l}
 a_i = 0 \quad \forall i \neq k
 a_k = x
\end{array}\right) =
\begin{array}{l}
 0 \implies p_j \circ q_k = 0 \text{ for } j \neq k \\
 (p_j \circ q_j)\(x) = p_j\(q_j\(x)) = p_j\left(\begin{array}{l}
 a_i = 0 \quad \forall i \neq j
 a_j = x
\end{array}\right) =
\end{array}
x \implies p_j \circ q_k = 1_{\mathbb{N}} \text{ for } j = k.

The universal algebra } (\mathbb{N}^I, \theta) \text{ satisfies the following property of universality:

Theorem 0.4. For every } A, \theta = \{\top, \bot, \theta_0\} \text{ an universal algebra of the some kind } \tau : \theta \to \mathbb{N}

as } (\mathbb{N}^I, \theta) \text{ and } u_i : A \to \mathbb{N} \text{ for every } i \in I \text{ morphisms between } (A, \theta) \text{ and } (\mathbb{N}, \Omega^*), \exists \text{ an unique } u : A \to \mathbb{N}^I \text{ morphisme between the universal algebras } (A, \theta) \text{ and } (\mathbb{N}^I, \theta) \text{ so that } p_j \circ u = u_j, \text{ for every } j \in I \text{ with } p_j : \mathbb{N}^I \to \mathbb{N}, \forall a = \{a_i\}_{i \in I} \in \mathbb{N}^I, p_j(a) = a_j \text{ the canonical projections morphisms between } (\mathbb{N}^I, \theta) \text{ and } (\mathbb{N}, \Omega^*).\n
Proof. The proof repeats the other one from the Theorem 1, step a).}

The property of universality establishes the universal algebra } (\mathbb{N}^I, \theta) \text{ until an isomorphism, which we can state by:

If } (P, \Omega^*) \text{ it is an universal algebra of the same kind as } (\mathbb{N}, \Omega^*) \text{ and } p'_i : P \to \mathbb{N} \text{ for every } i \in I \text{ a family of morphisms between } (P, \Omega^*) \text{ and } (\mathbb{N}, \Omega^*) \text{ so that for every universal algebra } (A, \theta) \text{ and every morphisms } u_i : A \to \mathbb{N}, \forall i \in I \text{ between } (A, \theta) \text{ and } (\mathbb{N}, \Omega^*) \text{ exists an unique morphisme } u : A \to P \text{ with } p'_i \circ u = u_i, \text{ for every } i \in I \text{ then it exists an unique isomorphism } f : P \to \mathbb{N}^I \text{ with } p_i \circ f = p'_i, \text{ for every } i \in I.

c) \text{ This third step contains the proof of the stated proposition (Problem 3).}

As } (\mathbb{N}^*, \Omega) \text{ with } \Omega = (V_d, l_0) \text{ is an universal algebra, in accordance with step a) it exists an universal algebra } ((\mathbb{N}^*)^I, \Omega) \text{ with } \Omega = \{\omega_1, \omega_0\} \text{ defined by:

\omega_1 : (\mathbb{N}^*)^I \to (\mathbb{N}^*)^I \text{ by every } a = \{a_i\}_{i \in I} \text{ and } b = \{b_i\}_{i \in I} \in (\mathbb{N}^*)^I,

\omega_1(a, b) = \{a_i V_d b_i\}_{i \in I}
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and

\[ \omega_0 : ((N^*)^I)^0 \rightarrow (N^*)^I \text{ by } \omega_0 (\{0\}) = \{e_i = 1\}_{i \in I} = e_{\omega_1}, \]

the canonical projections being certainly morphisms between \(( (N^*)^I, \Omega) \) and \(( N^*, \Omega) \).

As \(( N, \Omega') \) with \( \Omega' = \{V, \Psi_0\} \) is an universal algebra, in accordance with step b) it exists an universal algebra \(( N', \Omega' \) with \( \Omega' = \{\theta_1, \theta_0\} \) defined by:

\[ \theta_1 : (N')^2 \rightarrow N' \text{ by every } a = \{a_i\}_{i \in I}, b = \{b_i\}_{i \in I} \in N', \theta_1 (a, b) = \{a_i V b_i\}_{i \in I} \]

and

\[ \theta_0 : (N')^0 \rightarrow N' \text{ by } \theta_0 (\{0\}) = \{e_i = 0\}_{i \in I} = e_{\theta_1}, \]

The universal algebras \(( (N^*)^I, \Omega) \) and \(( N', \Omega') \) are of the same kind

\[
\begin{align*}
\omega_1 & \quad \omega_2 \\
2 & \quad 0 \quad = \quad \theta_1 & \quad \theta_0 \\
2 & \quad 0
\end{align*}
\]

We use the property of universality for universal algebra \(( N', \Omega' \) : an universal algebra \(( A, \Omega) \) can be \(( (N^*)^I, \Omega) \) because they are the same kind; the morphisms \( u_i : A \rightarrow N \) from the assumption will be \( \tilde{s}_i : (N^*)^I \rightarrow N^* \) by every \( a = \{a_i\}_{i \in I} \in (N^*)^I, \tilde{s}_i (a) = \tilde{s}_j (\{a_i\}_{i \in I}) = s (a_j) \iff \tilde{s}_j = s \circ p_j \) for every \( j \in I \) where \( s : N^* \rightarrow N \) is Smarandache's function and \( p_j : (N^*)^I \rightarrow N^* \) the canonical projections, morphisms between \(( (N^*)^I, \Omega) \) and \(( N^*, \Omega) \). As \( s \) is a morphisme between \(( N^*, \Omega) \) and \(( N, \Omega') \), \( \tilde{s}_j \) are morphisms (as a composition of morphisms) for every \( j \in I \). The assumptions of the property of universality being provided \( \implies \) exists an unique \( s : (N^*)^I \rightarrow N^I \) morphism between \(( (N^*)^I, \Omega) \) and \(( N^*, \Omega) \) so that \( p_j \circ s = \tilde{s}_j \iff p_j \circ s = S \circ p_j \) for every \( j \in I \). We finish the proof noticing that \( s \) is also surjection: \( p_j \circ S \) surjection (as a composition of surjections) \( \implies s \) surjection.

Remark: The proof of the step 3 can be done directly. As the universal algebras from the statement are built, we can define a correspondence \( s : (N^*)^I \rightarrow (N^*)^I \) by every \( a = \{a_i\}_{i \in I} \in (N^*)^I, s (a) = \{s (a_i)\}_{i \in I}, \) which is a function, then morphisme between the universal algebra of the same kind \(( (N^*)^I, \Omega) \) and \(( N^I, \Omega') \) and is also surjective, the required conditions being satisfied evidently.

The stated Problem finds a prolongation \( s \) of the Smarandache function \( S \) to more complex sets (for \( I = \{1\} \Rightarrow s = S \)). The properties of the function \( s \) for the limitation to \( N^* \) could bring new properties for the Smarandache function.
1. REFERENCES


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