Abstract: In this paper we determine all solutions of an exponential diophantine equations concerning the Smarandache square complementary function.

Key words: Smarandache square complementary function; exponential diophantine equations

For any positive integer $n$, let $SSC(n)$ denote the Smarandache square complementary function of $n$ (see [1]). In [3], Russo asked that solve the equation

$$SSC(n)^r + SSC(n)^{r-1} + \cdots + SSC(n) = n, \quad r > 1.$$  \hspace{1cm} (1)

In this paper we completely solve this problem as follows.

Theorem. All positive integer solutions $(n, r)$ of (1) are given by the following two cases.

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(i) \((n, r) = (363, 5)\).

(ii) \((n, r) = (ab^2, 2)\), where \(a\) and \(b\) are coprime positive integers satisfying \(a > 1, b > 1, a = b^2 - 1\) and \(a\) is square free.

The proof of our theorem needs the following lemma.

**Lemma** ([2]). The equation
\[
\frac{x^{r} - 1}{x - 1} = y^2, \quad x > 1, \quad y > 1, \quad r > 2
\]  
has only the positive integer solution \((x, y, r) = (3, 11, 5)\).

**Proof of Theorem.** Let \((n, r)\) be a positive integer solution of (1). Let \(x = \text{SSC}(n)\). Then from (1) we get
\[
x(x^{r-1} + \cdots + x + 1) = n, \quad r > 1.
\]  
Since \(r > 1\) we see from (3) that \(n > 1\).

It is a well known fact that \(n\) can be expressed as
\[
n = p_1^{\alpha_1} \cdots p_s^{\alpha_s} q_1^{\beta_1} \cdots q_l^{\beta_l},
\]  
where \(p_1, \ldots, p_s\) and \(q_1, \ldots, q_l\) are distinct primes, \(\alpha_1, \ldots, \alpha_s\) are odd positive integers and \(\beta_1, \ldots, \beta_l\) are even positive integers. We see from (4) that
\[
x = \text{SSC}(n) = p_1 \cdots p_s.
\]  
Since \(\gcd(x, x^{r-1} + \cdots + x + 1) = 1\), we get from (3), (4) and (5) that \(\alpha_1 = \cdots = \alpha_s = 1\) and
\[
\frac{x^{r-1}}{x-1} = x^{r-1} + \cdots + x + 1 = q_1^{\beta_1} \cdots q_l^{\beta_l}.
\]  
Since \(\beta_1, \ldots, \beta_l\) are even, let \(b^2 = q_1^{\beta_1} \cdots q_l^{\beta_l}\). Then \(b\) is a positive integer satisfying
\[ x^r - 1 = b^2. \tag{7} \]

By Lemma, if \( r > 2 \), then from (7) we get \((x, b, r) = (3, 11, 5)\). It implies that \((n, r) = (363, 5)\) by (4) and (15).

If \( r = 2 \), then we have
\[ x + 1 = b^2. \tag{8} \]

Let \( a = x \). By (4), (5) and (7), we obtain the case (ii) immediately. Thus, the theorem is proved.

References

