Some Properties of Smarandache Functions of the Type I

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We consider the construction of Smarandache functions of the type \( S_p(n) \) (\( p \in \mathbb{N}, p \text{ prime} \)) which are defined in [1] and [2] as follows:

\[
S_n : \mathbb{N}^* \rightarrow \mathbb{N}^* ; \quad S(k) = 1 ; \quad S(k) = \max_{1 \leq j < r} \{ S(i_j, k) \}
\]

for \( n = p_1^{i_1} p_2^{i_2} \ldots p_r^{i_r} \).

In this paper there are presented some properties of these functions. We shall study the monotonicity of each function \( S_n \) and also the monotonicity of some subsequences of the sequence \( \{ S_n \} \) \( n \in \mathbb{N}^* \).

1. Proposition. The function \( S_n \) is monotonous increasing for every positive integer \( n \).

Proof. The function \( S_1 \) is obviously monotonous increasing.

Let \( k_1 < k_2 \) where \( k_1, k_2 \in \mathbb{N}^* \). Supposing that \( n \) is a prime number and taking account that \( (S(k_2))! = \text{multiple of} \quad n_1 = \text{multiple of} \quad n_2 \),
it results that $S_n(k) \leq S_n(k)$, therefore $S_n$ is monotone increasing. Let
\[
S_n(k) = \max_{i \leq k} (S_i(k)) = S_{i_1}(k)
\]
\[
S_n(k) = \max_{i \leq k} (S_i(k)) = S_{i_1}(k)
\]
Because
\[
S(i, k) \leq S(i, k) \leq S(i, k)
\]
\[
S(i, k) \leq S(i, k) \leq S(i, k)
\]
it results that $S_n(k) \leq S_n(k)$ so $S_n$ is monotone increasing.

2. Proposition. The sequence of functions $(S_p(i))_{i \in \mathbb{N}}$ is monotone increasing, for every prime number $p$.

Proof. For any two numbers $i_1, i_2 \in \mathbb{N}^*$, $i_1 < i_2$ and for any $n \in \mathbb{N}^*$
we have:
\[
S_{i_1}(n) = S_{i_1}(n) \leq S_{i_2}(n) = S_{i_2}(n)
\]
\[
S_{i_1}(n) = S_{i_1}(n) \leq S_{i_2}(n) = S_{i_2}(n)
\]
Hence the sequence $(S_p(i))_{i \in \mathbb{N}^*}$ is monotone increasing for every prime number $p$.

3. Proposition. Let $p$ and $q$ two given prime numbers. If $p < q$ then

\[
S_p(k) < S_q(k), \quad k \in \mathbb{N}^*
\]

Proof. Let the sequence of coefficients (see (2)) $a_1^{(p)}, a_2^{(p)}, \ldots, a_{\infty}^{(p)}$.

Every $k \in \mathbb{N}^*$ can be uniquely written as
\[
k = t a_{1}^{(p)} + t a_{2}^{(p)} + \ldots + t a_{\infty}^{(p)}
\]
where $0 < t_i < p-1$ for $i = 1, \ldots, s-1$, and $0 < t_s < p$.

The procedure of passing from $k$ to $k+1$ in formula (1) is:

(C) $t_s$ is increasing with a unity.

(ii) if $t_s$ can not increase with a unity, then $t_{s-1}$ is increasing with a unity and $t_s = 0$

(iii) if neither $t_s$ nor $t_{s-1}$ are not increasing with a unity

then $t_{s-2}$ is increasing with a unity and $t_s = t_{s-1} = 0$

The procedure is continued in the same way until we obtain the expression of $k+1$.

Denoting $A_k(S) = S(p+1) - S(p)$ the leap of the function $S_p$ when we pass from $k$ to $k+1$ corresponding to the procedure described above. We find that

- in the case (i) $A_k(S) = p$

- in the case (ii) $A_k(S) = 0$

- in the case (iii) $A_k(S) = 0$

It is obviously seen that $S(1) = \sum_{k=1}^{n} A_k(S) + S(1)$.

Analogously we write

$S(n) = \sum_{k=1}^{n} A_k(S) + S(1)$

Taking into account that $S(1) = p < q = S(1)$ and using the procedure of passing from $k$ to $k+1$ we deduce that the number of leaps with zero value of $S$ is greater than the number of leaps with zero value of $S_q$, respectively the number of leaps with value $p$ of $S_p$ is less than the number of leaps of $S_q$ with value
\[
\sum_{k=1}^{n} \frac{A(S)}{p_k} + S(1) < \sum_{k=1}^{n} \frac{A(S)}{q_k} + S(1) \quad (2)
\]

Hence \( S(n) < S(n) \), \( n \in \mathbb{N}^* \).

As an example we give a table with \( S_2 \) and \( S_3 \) for \( 0 < n < 21 \):

\[
\begin{array}{cccccccccccccccccccc}
  k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
  \text{the leap} & 2 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\
  S_3(k) & 3 & 6 & 9 & 9 & 12 & 15 & 18 & 18 & 21 & 24 & 27 & 27 & 30 & 33 & 36 & 38 & 39 & 42 & 45 \\
\end{array}
\]

Hence \( S_2(k) < S_3(k) \) for \( k = 1, 2, \ldots, 20 \).

4. Remark. For any monotonous increasing sequence of prime numbers

\[ p_1 < p_2 < \ldots < p_n < \ldots \] it results that

\[ S_1 < S_{p_1} < S_{p_2} < \ldots < S_{p_n} < \ldots \]

If \( n = p_1^{i_1} p_2^{i_2} \ldots p_t^{i_t} \) and \( p_1 < p_2 < \ldots < p_t \) then

\[ S_n(k) = \max_{i \leq j \leq k} (S_{p_j}(k)) = S_{p_1}(k) = S_{p_{i_1}}(k) \]

5. Proposition. If \( p \) and \( q \) are prime numbers and \( p, i < q \) then \( S_{p} < S_{q} \).

Proof. Because \( p, i < q \) it results

\[ S_{p}(1) \leq p, i < q = S_{q}(1) \quad (3) \]

and \( S_{p}(k) = S_{p}(1) \leq p, i \leq S_{q}(k) \).

From (3) passing from \( k \) to \( k+1 \), we deduce

\[ A_k(S_{p}) \leq i A_k(S_{p}) \quad (4) \]

Taking into account the proposition 3. from (4) it results that when we pass from \( k \) to \( k+1 \) we obtain
\[ \sigma_k(p) \leq \sigma_k(q) \leq 1, p < q \text{ and } \sum_{k=1}^{n} \sigma_k(p) \leq \sum_{k=1}^{n} \sigma_k(q) \quad (6) \]

Because we have

\[ \sigma_p(n) = \sigma_p(1) + \sum_{k=1}^{n} \sigma_k(p) \leq \sigma_q(1) + \sum_{k=1}^{n} \sigma_k(q) \]

and

\[ \sigma_q(n) = \sigma_q(1) + \sum_{k=1}^{n} \sigma_k(q) \]

from (3) and (5) it results

\[ \sigma_p(n) \leq \sigma_q(n), \quad n \in \mathbb{N}^* \]

8. Proposition. If \( p \) is a prime number then \( \sigma_p(n) < \sigma_p(n) \) for every \( n < p \).

Proof. If \( n \) is a prime number from \( n < p \), using the proposition 3 it results \( \sigma_n(k) < \sigma_p(k) \) for \( k \in \mathbb{N}^* \). If \( n \) is a composed, that is \( n = p_1^{i_1} \ldots p_i^{i_i} \) then \( \sigma_n(k) = \max_{1 \leq i \leq j} (\sigma_i(k)) = \sigma_p(k) \).

Because \( n < p \) it results \( p_r < p \) and using the proposition 5 and knowing that \( i_r p_r \leq p_r < p \) it results that \( \sigma_r(k) \leq \sigma_p(k) \)

therefore \( \sigma_r(k) < \sigma_p(k) \).

References
