SOME REMARKS CONCERNING THE DISTRIBUTION
OF THE SMARANDACHE FUNCTION

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The Smarandache function is a numerical function \( S:N^* \rightarrow N^* \) \( S(k) \) representing the smallest natural number \( n \) such that \( n! \) is divisible by \( k \). From the definition it results that \( S(1)=1 \).

I will refer for the beginning the following problem:

"Let \( k \) be a rational number, \( 0 < k \leq 1 \). Does the diophantine equation \( \frac{S(n)}{n} = k \) has always solutions? Find all \( k \) such that the equation has an infinite number of solutions in \( N^* \)" from "Smarandache Function Journal".

I intend to prove that equation hasn't always solutions and case that there are an infinite number of solutions is when \( k = \frac{1}{r} \), \( r \in N^* \), \( k \in Q \) and \( 0 < k \leq 1 \) \( \Rightarrow \) there are two relatively prime non negative integers \( p \) and \( q \) such that \( k = \frac{p}{q} \), \( p,q \in N^* \), \( 0 < q \leq p \). Let \( n \) be a solution of the equation \( \frac{S(n)}{n} = k \). Then \( \frac{S(n)}{n} = \frac{p}{q} \), (1). Let \( d \) be a highest common divisor of \( n \) and \( S(n) : d = (n, S(n)) \). The fact that \( p \) and \( q \) are relatively prime and (1) implies that \( S(n) = qd \), \( n = pd \Rightarrow S(pd) = qd \) (*).

This equality gives us the following result: \((qd)! \) is divisible by \( pd \Rightarrow [(qd - 1)!-q] \) is divisible by \( p \). But \( p \) and \( q \) are relatively prime integers, so \((qd-1)! \) is divisible by \( p \). Then \( S(p) \leq qd - 1 \).

I prove that \( S(p) \geq (q - 1)d \).

If we suppose against all reason that \( S(p) < (q - 1)d \), it means \([ (q - 1)d - 1]! \) is divisible by \( p \). Then \( (pd)! | [(q - 1)d]! \) because \( d | (q - 1)d \), so \( S(pd) \leq (q - 1)d \). This is contradiction with the fact that \( S(pd) = qd > (q - 1)d \). We have the following inequalities:

\[(q - 1)d \leq S(p) \leq qd - 1.\]

For \( q \geq 2 \) we have from the first inequality \( d \leq \frac{S(p)}{q-1} \) and from the second \( \frac{S(p+1)}{q} \leq d \), so

\[\frac{S(p+1)}{q} \leq d \leq \frac{S(p)}{q-1}.\]
For \( k = \frac{q}{p} \), \( q \geq 2 \), the equation has solutions if and only if there is a natural number between \( \frac{S(p-1)}{q} \) and \( \frac{S(p)}{q-1} \). If there isn’t such a number, then the equation hasn’t solutions. However, if there is a number \( d \) with \( \frac{S(p-1)}{q} \leq d \leq \frac{S(p)}{q-1} \), this doesn’t mean that the equation has solutions. This condition is necessary but not sufficient for the equation to have solutions.

For example:

a) \( k = \frac{4}{5} \), \( q = 4 \), \( p = 5 \) \( \Rightarrow \frac{S(p-1)}{q} = \frac{6}{4} = 1.5 \), \( \frac{S(p)}{q-1} = \frac{5}{3} \). In this case the equation hasn’t solutions.

b) \( k = \frac{3}{10} \), \( q = 3 \), \( p = 10 \); \( S(10) = 5 \), \( \frac{6}{3} = 2 \leq d \leq \frac{5}{2} \). If the equation has solutions, then we must have \( d = 2 \), \( n = dp = 20 \), \( S(n) = dq = 6 \). But \( S(20) = 5 \).

This is a contradiction. So there are no solutions for \( k = \frac{3}{10} \).

We can have more than natural numbers between \( \frac{S(p+1)}{q} \) and \( \frac{S(p)}{q-1} \). For example:

\[ k = \frac{3}{29}, \ q = 3, \ p = 29, \ \frac{S(p+1)}{q} = 10, \ \frac{S(p)}{q-1} = 14.5. \]

We prove that the equation \( \frac{S(n)}{n} = k \) hasn’t always solutions.

If \( q \geq 2 \) then the number of solutions is equal with the number of values of \( d \) that verify relation (*). But \( d \) can be a nonnegative integer between \( \frac{S(p+1)}{q} \) and \( \frac{S(p)}{q-1} \), so \( d \) can take only a finite set of values. This means that the equation has no solutions or it has only a finite number of solutions.

We study now case \( k = \frac{1}{p}, \ p \in \mathbb{N^*} \). In this case the equation has an infinite number of solutions. Let \( p_0 \) be a prime number such that \( p < p_0 \) and \( n = pp_0 \). We have \( S(n) = S(pp_0) = p \), so \( S(n) = p_0 \), \( \frac{S(n)}{n} = \frac{p_0}{p p_0} = \frac{1}{p} \), so the equation has an infinite number of solutions.

I will refer now to another problem concerning the ratio \( \frac{S(n)}{n} \) "Is there an infinity of natural numbers such that \( 0 < \left\{ \frac{x}{S(x)} \right\} \leq \left\{ \frac{S(x)}{x} \right\} \)" from the same journal.

I will prove that the only number \( x \) that verifies the inequalities is \( x = 9 \) : \( S(9) = 6, \)
\[
\frac{x}{S(x)} = \frac{9}{6} = \frac{3}{2}, \quad \left\{ \frac{x}{S(x)} \right\} = \left\{ \frac{9}{6} \right\} = \frac{1}{2} \quad \text{and} \quad 0 < \frac{1}{2} < \frac{2}{3}, \quad \text{so } x = 9 \text{ verifies } 0 < \left\{ \frac{x}{S(x)} \right\} < \left\{ \frac{S(x)}{x} \right\}. \]

Let \( x = p_1^{a_1} \ldots p_n^{a_n} \) be the standard form of \( x \).

\( S(x) = \max S(p_k^{a_k}) \). We put \( S(x) = S(p^a) \), where \( p^a \) is one of \( p_1^{a_1} \ldots p_n^{a_n} \) such that \( S(p^a) = \max_{1 \leq k \leq n} S(p_k^{a_k}) \).
\[ \frac{x}{S(x)} \] can take one of the following values: \( \frac{1}{S(x)} \), \( \frac{2}{S(x)} \), ..., \( \frac{S(x)-1}{S(x)} \) because
\[
0 < \left( \frac{x}{S(x)} \right) < \frac{S(x)}{x} \quad (\text{We have } S(x) \leq x, \text{ so } \frac{S(x)}{x} \leq 1 \text{ and } \left( \frac{S(x)}{x} \right) \leq S(x) \text{ ). This means}
\]
\[
\frac{S(x)}{x} \geq \frac{1}{S(x)} \Rightarrow S(p^a)^2 > x \geq p^a. \quad (2)
\]

But \( (ap)! = 1 \cdot 2 \cdot ... \cdot (p-1) ...(2p)...(ap) \) is divisible by \( p^a \), so \( ap \geq S(p^a) \). From this last inequality and (2) it follows that \( \alpha^2 p^2 > p^2 \). We have three cases:

I. \( \alpha = 1 \). In this case \( S(x) = S(p) = p \), \( x \) is divisible by \( p \), so \( \frac{x}{p} \in \mathbb{Z} \). This is a contradiction.

There are no solutions for \( \alpha = 1 \).

II. \( \alpha = 2 \). In this case \( S(x) = S(p^2) = 2p \), because \( p \) is a prime number and \( (2p)! = 1 \cdot 2 \cdot ... \cdot (p-1) ...(2p) \), so \( S(p^2) = 2p \).

But \( \left\{ \frac{px}{2} \right\} = \left\{ \frac{1}{2} \right\} \). This means \( \left\{ \frac{px}{2} \right\} = \frac{1}{2} \Rightarrow \frac{2}{px} < \frac{2}{p} < 4 \); \( p \) is a prime number \( \Rightarrow p \in \{2,3\} \).

If \( p = 2 \) and \( px_1 < 4 \Rightarrow x_1 = 1 \), but \( x = 4 \) isn’t a solution of the equation: \( S(4) = 4 \) and \( \left\{ \frac{4}{4} \right\} = 0 \).

If \( p = 3 \) and \( px_1 < 4 \Rightarrow x_1 = 1 \), so \( x = p^2 = 9 \) is a solution of equation.

III. \( \alpha = 3 \). We have \( \alpha^2 p^2 > p^a \Rightarrow \alpha^2 > p^{a-1} \).

For \( \alpha \geq 8 \) we prove that we have \( p^{a-2} > p^2 \), \( (\forall) p \in \mathbb{N}^*, p \geq 2 \).

We prove by induction that \( 2^{n-1} > (n-1)^2 \).

\[
2^{n-1} = 2 \cdot 2^{n-2} > n^2 - 2 \cdot n + 1 = n^2 - 2n + n^2 - 8n + 8 = n^2 + 2n + 1 = (n-1)^2, \quad \text{because } n \geq 8.
\]

We proved that \( p^{a-2} \geq 2^{a-1} \geq 2^a \), for any \( \alpha \geq 8, p \in \mathbb{N}^*, \ p \geq 2 \).

We have to study the case \( \alpha \in \{3,4,5,6,7\} \).

a) \( \alpha = 3 \Rightarrow p \in \{2,3,5,7\} \), because \( p \) is a prime number.

If \( p = 2 \) then \( S(x) = S(2^3) = 4 \). But \( x \) is divisible by 8, so \( \left\{ \frac{x}{4} \right\} = \left\{ \frac{x}{S(x)} \right\} \), so \( x = 4 \) cannot be a solution of the inequation.

If \( p = 3 \Rightarrow S(x) = S(3^3) = 9 \). But \( x \) is divisible by 9, so \( \left\{ \frac{x}{9} \right\} = \left\{ \frac{x}{S(x)} \right\} \), so \( x = 9 \) cannot be a solution of the inequation.

If \( p = 5 \Rightarrow S(x) = S(5^3) = 15 \); \( \left\{ \frac{x}{S(x)} \right\} = \left\{ \frac{x}{5^3} \right\} = \left\{ \frac{x}{x} \right\} = 0 \), so \( x = 5^3 \cdot x_1 \), \( x_1 \in \mathbb{N}^* \), \( (5,x_1) = 1 \).
We have $0 < \left\{ \frac{5^2 \cdot x}{3} \right\} < \left\{ \frac{3}{5^2 \cdot x} \right\}$. This first inequality implies $\left\{ \frac{5^2 \cdot x}{3} \right\} \notin \left\{ \frac{1}{3}, \frac{2}{3} \right\}$, so $\frac{1}{3} < \frac{3}{5^2 \cdot x_1}$, $\Rightarrow 5^2 \cdot x_1 < 9$, but this is impossible.

If $p=7 \Rightarrow S(x)=S(3^7)=21, x=7^3 \cdot x_1, (7,x_1)=1, x_1 \in N^*.$

We have $0 < \left\{ \frac{x}{S(x)} \right\} < \left\{ \frac{S(x)}{x} \right\} \Rightarrow 0 < \left\{ \frac{7^2 \cdot x}{3} \right\} < \frac{3}{7^2 \cdot x_1}$. But $0 < \left\{ \frac{7^2 \cdot x}{3} \right\}$ implies $\left\{ \frac{7^2 \cdot x}{3} \right\} \in \left\{ \frac{1}{3}, \frac{2}{3} \right\}$.

W have $\frac{1}{3} < \left\{ \frac{7^2 \cdot x}{3} \right\} \Rightarrow 7^2 \cdot x_1 < 9$, but is impossible.

b) $\alpha=4 : 16 \Rightarrow p \in \{2,3\}$.

If $p=2 \Rightarrow S(x)=S(x^2)=6, x=16 \cdot x_1, x_1 \in N^*, (2,x_1)=1, 0 < \left\{ \frac{x}{S(x)} \right\} < \frac{S(x)}{x} \Rightarrow 0 < \left\{ \frac{8x_1}{3} \right\} < \frac{3}{8x_1}$.

$0 < \left\{ \frac{8x_1}{3} \right\} \Rightarrow x_1=1 \Rightarrow x=16$.

But $\frac{S(x)}{x} = \frac{6}{16} = \frac{3}{8}; \left\{ \frac{x}{S(x)} \right\} = \left\{ \frac{16}{6} \right\} = \left\{ \frac{8}{3} \right\} = \left\{ \frac{2}{3} \right\} = \left\{ \frac{2}{3} \right\} > \frac{3}{8}$, so the inequality isn’t verified.

If $p=3 \Rightarrow S(x)=S(3^4)=9, x=3^4 \cdot x_1, (3,x_1)=1 \Rightarrow 9|x \Rightarrow \frac{x}{S(x)} = 0$, so the inequality isn’t verified.

For $\alpha=\{5,6,7\}$, the only natural number $p>1$ that verifies the inequality $\alpha^2 > p^{\alpha-2}$ is 2:

$\alpha=5 : 25 > p^3 \Rightarrow p=2$

$\alpha=6 : 36 > p^4 \Rightarrow p=2$

$\alpha=7 : 49 > p$

In every case $x=2^\alpha \cdot x_1, x_1 \in N^*, (x_1,2)=1, \text{ and } S(x_1) \leq S(2^\alpha)$.

But $S(2^5) = S(2^6) = S(2^7) 8$, so $S(x) = 8$ But $x$ is divisible by 8, so $\left\{ \frac{x}{S(x)} \right\} = 0$ so the inequality isn’t verified because $0 < \left\{ \frac{x}{S(x)} \right\}$. We found that there is only $x=9$ to verify the inequality $0 < \left\{ \frac{x}{S(x)} \right\} < \left\{ \frac{S(x)}{x} \right\}$.

I try to study some diophantine equations proposed in "Smarandache Function Journal".

1) I study the equation $S(mx)=mS(x), m \geq 2$ and $x$ is a natural number.
Let \( x \) be a solution of the equation.

We have \( S(x)! \) is divisible by \( x \). It is known that among \( m \) consecutive numbers, one is divisible by \( m \), so \( (S(x)+1)(S(x)+2)\cdots(S(x)+m) \) is divisible by \( (mx) \). We know that \( S(mx) \) is the smallest natural number such that \( S(mx)! \) is divisible by \( (mx) \) and this implies \( S(mx) \leq S(x)+m \). But \( S(mx)=mS(x) \), so \( mS(x) \leq S(x)+m \Rightarrow mS(x)-S(x)-m-1 \leq (m-1)(S(x)-1) \leq 1 \). We have several cases:

If \( m=1 \) then the equation becomes \( S(X)=S(x) \), so any natural number is a solution of the equation.

If \( m=2 \), we have \( S(x) \in \{1,2\} \) implies \( x \in \{1,2\} \). We conclude that if \( m=1 \) then any natural number is a solution of the equation; if \( m=2 \) then \( x=1 \) and \( x=2 \) are only solution and if \( m \geq 3 \) the only solution of the equation is \( x=1 \).

2) Another equation is \( S(x^y)=y^x \), \( x, y \) are natural numbers.

Let \( (x,y) \) be a solution of the equation.

\((xy)! = 1 \cdots x(x-1) \cdots (2x-1) \cdots (yx)! \) implies \( S(x^y) \leq yx \), so \( yx \leq yx \) because \( S(x^y)=y^x \).

But \( y \geq 1 \), so \( y^{x-1} \leq x \).

If \( x=1 \) then equation becomes \( S(1) = y \), so \( y=1 \), so \( x=y=1 \) is a solution of the equation.

If \( x \geq 2 \) then \( x \geq 2^{x-1} \). But the only natural numbers that verify this inequality are \( x=y=2 \):

\( x=y=2 \) verifies the equation, so \( x=y=2 \) is a solution of the equation.

For \( x \geq 3 \) we prove that \( x < 2^{x-1} \). We make the proof by induction.

If \( x=3 \) : \( 3 < 2^{3-1} = 4 \).

We suppose that \( k < 2^{k-1} \) and we prove that \( k+1 < 2^k \). We have \( 2^k = 2 \cdot 2^{k-1} > 2 \cdot k = k + k = k + 1 \), so the inequality is established and there are no other solutions then \( x=y=1 \) and \( x=y=2 \).

3) I will prove that for any \( m,n \) natural numbers, if \( m>1 \) then the equation \( S(x^n)=x^m \) has no solution or it has a finite number of solutions, and for \( m=1 \) the equation has an infinite number of solutions.

I prove that \( S(x^n) \leq nx \). But \( x^m = S(x^n) \), so \( x^m \leq nx \).

For \( m \geq 2 \) we have \( x^{m-1} \leq x \). If \( m=2 \) then \( x \leq n \), and if \( m \geq 3 \) then \( x \leq \sqrt[n]{n} \), so \( x \) can take only a finite number of values, so the equation can have only a finite number of solutions or it has no solutions.

We notice that \( x=1 \) is a solution of the equation for any \( m,n \) natural numbers.
If the equation has a solution different of 1, we must have \( x^{m} = S(x^n) \leq n \), so \( m \leq n \).

If \( m = n \), the equation becomes \( x^{m} = S(x^n) \), so \( x^n \) is a prime number or \( x^n = 4 \), so \( n = 1 \) and any prime number as well as \( x = 4 \) is a solution of the equation, or \( n = 2 \) and the only solutions are \( x = 1 \) and \( x = 2 \).

For \( m = 1 \) and \( n \geq 1 \), we prove that the equations \( S(x^m) = x \), \( x \in \mathbb{N}^* \) has an infinite number of solutions. Let be a prime number, \( p > n \). We prove that \( (np) \) is a solution of the equation, that is \( S((np)^n) = np \).

\( n < p \) and \( p \) is a prime number, so \( n \) and \( p \) are relatively prime numbers.

\( n < p \) implies:

\[(np)! = 1 \cdot 2 \cdot \ldots \cdot n(n+1) \cdot \ldots \cdot (2n) \cdot \ldots \cdot (pn) \text{ is divisible by } n^n.\]

\[(np)! = 1 \cdot 2 \cdot \ldots \cdot p(p+1) \cdot \ldots \cdot (2p) \cdot \ldots \cdot (pn) \text{ is divisible by } p^n.\]

But \( p \) and \( n \) are relatively prime numbers, so \( (np)! \) is divisible by \( (np)^n \).

If we suppose that \( S((np)^n) < np \), then we find that \( (np-1)! \) is a divisible by \( (np)^n \), so \( (np-1)! \) is divisible by \( p^n(3) \). But the exponent of \( p \) in the standard form of \( p \) in the standard form of \( (np-1)! \) is:

\[E = \left[ \frac{np-1}{p} \right] + \left[ \frac{np-1}{p^2} \right] + \ldots\]

But \( p > n \), so \( p^2 > np > np-1 \). This implies:

\[\left[ \frac{np-1}{p^k} \right] = 0 \text{, for any } k \geq 2.\]

We have:

\[E = \left[ \frac{np-1}{p} \right] = n - 1.\]

This means \( (np-1)! \) is divisible by \( p^{n-1} \), but isn’t divisible by \( p^n \), so this is a contradiction with (3). We proved that \( S((np)n) = np \), so the equation \( S(x^n) = x \) has an infinite number of solutions for any natural number \( n \).

REFERENCE


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