Abstract

In this paper four Smarandache product sequences have been studied: Smarandache Square product sequence, Smarandache Cubic product sequence, Smarandache Factorial product sequence and Smarandache Palprime product sequence. In particular the number of primes, the convergence value for Smarandache Series, Smarandache Continued Fractions, Smarandache Infinite product of the mentioned sequences has been calculated utilizing the Ubasic software package. Moreover for the first time the notion of Smarandache Continued Radicals has been introduced. One conjecture about the number of primes contained in these sequences and new questions are posed too.

Introduction

In [1] Iacobescu describes the so called Smarandache U-product sequence. Let $u_n \geq 1$, be a positive integer sequence. Then a U-sequence is defined as follows:

$$U_n = 1 + u_1 \cdot u_2 \cdot \ldots \cdot u_n$$

In this paper differently from [1], we will call this sequence a U-sequence of the first kind because we will introduce for the first time a U-sequence of the second kind defined as follows:

$$U_n = |1 - u_1 \cdot u_2 \cdot \ldots \cdot u_n|$$

In this paper we will discuss about the “Square product”, “Cubic product”, “Factorial product” and “Primorial product” sequences. In particular we will analyze the question posed by Iacobescu in [1] on the number of primes contained in those sequences. We will also analyze the convergence values of the Smarandache Series [2], Infinite product [3], Simple Continued Fractions [4] of the four sequences. Moreover for the first time we will introduce the notion of Smarandache Continued Radicals and we will analyse the convergence of sequences reported above.
Sequences details

**Smarandache square product sequence of the first and second kind.**

In this case the sequence $u_n$ is given by:

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144......

that is the square of $n$. The first 20 terms of the sequence $U_n$ $(1 \leq n \leq 20)$ both the first and second kind are reported in the table below:

<table>
<thead>
<tr>
<th>Smarandache Square product sequence (first kind)</th>
<th>Smarandache Square product sequence (second kind)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>13</td>
</tr>
<tr>
<td>37</td>
<td>39</td>
</tr>
<tr>
<td>577</td>
<td>577</td>
</tr>
<tr>
<td>14401</td>
<td>14401</td>
</tr>
<tr>
<td>518401</td>
<td>518401</td>
</tr>
<tr>
<td>25401601</td>
<td>25401599</td>
</tr>
<tr>
<td>1625702401</td>
<td>1625702399</td>
</tr>
<tr>
<td>131681894401</td>
<td>131681894399</td>
</tr>
<tr>
<td>13168189440001</td>
<td>1316818943999</td>
</tr>
<tr>
<td>15933509222240001</td>
<td>159335092223999</td>
</tr>
<tr>
<td>229442532802560001</td>
<td>229442532802559999</td>
</tr>
<tr>
<td>38775788043632640001</td>
<td>38775788043632639999</td>
</tr>
<tr>
<td>7600054456551997440001</td>
<td>7600054456551997439999</td>
</tr>
<tr>
<td>1710012252724199424000001</td>
<td>171001225272419942399999</td>
</tr>
<tr>
<td>43776313669739505254400001</td>
<td>43776313669739505254399999</td>
</tr>
<tr>
<td>126513546505547170185216000001</td>
<td>12651354650554717018521599999</td>
</tr>
<tr>
<td>40990389067797283140009984000001</td>
<td>40990389067797283140009983999999</td>
</tr>
<tr>
<td>14797530453474819213543604224000001</td>
<td>14797530453474819213543604223999999</td>
</tr>
<tr>
<td>5919012181389927685417441689600000001</td>
<td>59190121813899276854174416895999999999</td>
</tr>
</tbody>
</table>

**Smarandache cubic product sequence of the first and second kind.**

In this case the sequence $u_n$ is given by:

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728......

that is the cube of $n$. Here the first 17 terms for the sequence $U_n$ of the first and second kind.
\[
\begin{array}{|c|c|}
\hline
\text{Smarandache Cubic product sequence (first kind)} & \text{Smarandache Cubic product sequence (second kind)} \\
\hline
2 & 0 \\
9 & 7 \\
217 & 215 \\
13825 & 13823 \\
1728001 & 1727999 \\
373248001 & 372247999 \\
128024064001 & 128024063999 \\
65548320768001 & 65548320767999 \\
477847258398720001 & 477847258398719999 \\
47784725839872000001 & 47784725839871999999 \\
636014700928696932000001 & 636014700928696931999999 \\
109903340320478724096000001 & 1099033403204787240959999999 \\
241457638684091756838912000001 & 24145763868409175683891199999999 \\
6625597605491477807659745230000001 & 662559760549147780765974527999999999 \\
2236139191853373760085164032000000001 & 223613919185337376008516403199999999999 \\
9159226129831418921308831875072000000001 & 9159226129831418921308831875071999999999999 \\
44999277975861761160392091002228360000000001 & 449992779758617611603920910022287359999999999999 \\
\hline
\end{array}
\]

\textit{o Smarandache factorial product sequence of the first and second kind.}

In this case the sequence \( u_n \) is given by:

\[ 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \ldots \]

that is the factorial of \( n \). The first 13 terms of the \( U_n \) sequence of the first and second kind follow.

\[
\begin{array}{|c|c|}
\hline
\text{Smarandache Factorial product sequence (first kind)} & \text{Smarandache Factorial product sequence (second kind)} \\
\hline
2 & 0 \\
3 & 1 \\
13 & 11 \\
289 & 287 \\
34561 & 34559 \\
34883201 & 34883199 \\
125411328001 & 125411327999 \\
5056584744960001 & 50565847449599999 \\
18349334722510848000001 & 183493347225108479999999 \\
6658606584104756822400000001 & 665860658410475682239999999999 \\
26579026729639194868109496320000000001 & 26579026729639194868109496319999999999999 \\
12731396329939944674959771247411200000000001 & 127313963299399446749597712474111999999999999999 \\
79278669759579679560737708649087148855296000000000001 & 79278669759579679560737086400871488552959999999999999999 \\
\hline
\end{array}
\]
In this case the sequence $u_n$ is given by:

$$2, 3, 5, 7, 11, 101, 121, 131, 151, 181, 191, 313, 353, 353, 373, \ldots$$

that is the sequence of palindromic primes. Below the first 17 terms of the $U_n$ sequence of the first and second kind.

<table>
<thead>
<tr>
<th>Smarandache Palprime product sequence (first kind)</th>
<th>Smarandache Palprime product sequence (second kind)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>31</td>
<td>29</td>
</tr>
<tr>
<td>211</td>
<td>209</td>
</tr>
<tr>
<td>2311</td>
<td>2309</td>
</tr>
<tr>
<td>233311</td>
<td>233309</td>
</tr>
<tr>
<td>28320511</td>
<td>28320509</td>
</tr>
<tr>
<td>3698196811</td>
<td>3698196809</td>
</tr>
<tr>
<td>558427718311</td>
<td>558427718309</td>
</tr>
<tr>
<td>101075417014111</td>
<td>101075417014109</td>
</tr>
<tr>
<td>1930540469695011</td>
<td>1930540469695009</td>
</tr>
<tr>
<td>6042591655354538131</td>
<td>6042591655354538129</td>
</tr>
<tr>
<td>2133034854340151959891</td>
<td>2133034854340151959889</td>
</tr>
<tr>
<td>79562200666876681038971</td>
<td>79562200666876681038969</td>
</tr>
<tr>
<td>304723226256179768837925511</td>
<td>304723226256179768837925509</td>
</tr>
<tr>
<td>221533785488242691945171845771</td>
<td>221533785488242691945171845769</td>
</tr>
<tr>
<td>167701075614599717802495087247891</td>
<td>167701075614599717802495087247889</td>
</tr>
</tbody>
</table>

Results

For all above sequences the following questions have been studied:

1. How many terms are prime?
2. Is the Smarandache Series convergent?
3. Is the Smarandache Infinite product convergent?
4. Is the Smarandache Simple Continued Fractions convergent?
5. Is the Smarandache Continued Radicals convergent?

For this purpose the software package Ubasic Rev. 9 has been utilized. In particular for the item n. 1, a strong pseudoprime test code has been written [5]. Moreover, as already mentioned above, the item 5 has been introduced for the first time; a Smarandache Continued Radicals is defined as follows:

$$\sqrt{a(1)} + \sqrt{a(2)} + \sqrt{a(3)} + \sqrt{a(4)} + \ldots$$
where a(n) is the nth term of a Smarandache sequence. Here below a summary table of the obtained results:

<table>
<thead>
<tr>
<th># Primes</th>
<th>SS cv</th>
<th>SIP cv</th>
<th>S S C F cv</th>
<th>SCR cv</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square 1st kind</td>
<td>12.456 = 0.026</td>
<td>0.7288153796...</td>
<td>2.1989247812...</td>
<td>2.3666079803...</td>
</tr>
<tr>
<td>Square 2nd kind</td>
<td>1.463 = 0.0021</td>
<td>∞</td>
<td>0.3301888340...</td>
<td>1.8143775546...</td>
</tr>
<tr>
<td>Cubic 1st kind</td>
<td>@</td>
<td>0.6157923201...</td>
<td>2.1110542477...</td>
<td>2.6904314681...</td>
</tr>
<tr>
<td>Cubic 2nd kind</td>
<td>@</td>
<td>∞</td>
<td>0.1427622842...</td>
<td>2.2446138067...</td>
</tr>
<tr>
<td>Factorial 1st kind</td>
<td>5.70 = 0.071</td>
<td>0.9137455924...</td>
<td>2.3250021620...</td>
<td>2.2332152218...</td>
</tr>
<tr>
<td>Factorial 2nd kind</td>
<td>2.66 = 0.033</td>
<td>∞</td>
<td>0.9166908563...</td>
<td>1.6117607295...</td>
</tr>
<tr>
<td>Palprime 1st kind</td>
<td>10.363 = 0.027</td>
<td>0.5136249121...</td>
<td>3.142019345...</td>
<td>2.5932060878...</td>
</tr>
<tr>
<td>Palprime 2nd kind</td>
<td>9.363 = 0.024</td>
<td>1.2397048573...</td>
<td>1.1986303614...</td>
<td>2.1032632883...</td>
</tr>
</tbody>
</table>

Legend:

- # primes (Number of primes/number of sequence terms checked)
- SS cv (Smarandache Series convergence value)
- SIP cv (Smarandache Infinite Product convergence value)
- S S C F cv (Smarandache Simple Continued Fractions convergence value)
- SCR cv (Smarandache Continued Radicals convergence value)
- @ (This sequence contain only one prime as proved by M. Le and K. Wu [6])

About the items 2, 3, 4 and 5 according to these results the answer is: yes, all the analyzed sequences converge except the Smarandache Series and the Smarandache Infinite product for the square product (2nd kind), cubic product (2nd kind) and factorial product (2nd kind). In particular notice the nice result obtained with the convergence of Smarandache Simple Continued Fractions of Smarandache palprime product sequence of the first kind.

The value of convergence is roughly π with the first two decimal digits correct.

\[ \pi \approx \frac{1}{7} + \frac{1}{31 + \frac{1}{211 + \frac{1}{2311 + \frac{1}{233311 + \ldots}}}} \]

Analogously for the cubic product sequence of the second kind the simple continued fraction converge roughly to \( \pi - 3 \), while for the factorial product sequence of the second kind the continued radical converge roughly (two first decimal digits correct) to the golden ratio \( \phi \), that is:

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About the item 1, the following table reports the values of n in the sequence that generate a strong pseudoprime number and its digit’s number.

<table>
<thead>
<tr>
<th></th>
<th>n</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square 1st kind</td>
<td>1/2/3/4/5/9/10/11/1324/65/76</td>
<td>1/1/2/3/5/12/14/16/20/48/182/223</td>
</tr>
<tr>
<td>Square 2nd kind</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Cubic 1st kind</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Cubic 2nd kind</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Factorial 1st kind</td>
<td>1/2/3/7/14</td>
<td>1/1/2/125/65</td>
</tr>
<tr>
<td>Factorial 2nd kind</td>
<td>3/7</td>
<td>2/12.</td>
</tr>
<tr>
<td>Palprime 1st kind</td>
<td>1/2/3/4/5/7/10/19/57/234</td>
<td>1/1/2/3/4/8/15/59/198/1208</td>
</tr>
<tr>
<td>Palprime 2nd kind</td>
<td>2/3/4/5/7/10/19/57/234</td>
<td>1/2/3/4/8/15/59/198/1208</td>
</tr>
</tbody>
</table>

Please note that the primes in the sequence of palprime of the first and second kind generate pairs of twin primes. The first ones follow:

(3,5) (5,7) (29,31) (209,211) (2309,2311) (28230509,28230511) (101075417014109,101075417014111) .......

Due to the fact that the percentage of primes found is very small and that according to Prime Number Theorem, the probability that a randomly chosen number of size n is prime decreases as 1/d (where d is the number of digits of n) we are enough confident to pose the following conjecture:

- The number of primes contained in the Smarandache Square product sequence (1st and 2nd kind), Smarandache Factorial product sequence (1st and 2nd kind) and Smarandache Palprime product sequence (1st and 2nd kind) is finite.
New Questions

- Is there any Smarandache sequence whose SS, SIP, SSCF and SCR converge to some known mathematical constants?

- Are all the estimated convergence values irrational or transcendental?

- Is there for each prime inside the Smarandache Palprime product sequence of the second kind the correspondent twin prime in the Smarandache Palprime product sequence of the first kind?

- Are there any two Smarandache sequences \( a(n) \) and \( b(n) \) whose Smarandache Infinite Product ratio converge to some value \( k \) different from zero?

\[
\lim_{n \to \infty} \frac{1}{\prod_{n} a(n)} = k
\]

\[
\lim_{n \to \infty} \frac{1}{\prod_{n} b(n)}
\]

- Is there any Smarandache sequence \( a(n) \) such that:

\[
\lim_{n \to \infty} e^{\frac{1}{n^2 a(n)}} \approx \pi
\]

- For the four sequences of first kind \( a(n) \), study:

\[
\lim_{n \to \infty} \sum_{n} a(n) = \sum_{n} R(a(n))
\]

where \( R(a(n)) \) is the reverse of \( a(n) \). (For example if \( a(n) = 17 \) then \( R(a(n)) = 71 \) and so on).
References