THE AVERAGE VALUE OF THE SMARANDACHE FUNCTION

Steven R. Finch
MathSoft Inc.
101 Main Street
Cambridge, MA, USA 02142
sfinch@mathsoft.com

Given a positive integer \( n \), let \( P(n) \) denote the largest prime factor of \( n \) and \( S(n) \) denote the smallest integer \( m \) such that \( n \) divides \( m! \).

The function \( S(n) \) is known as the Smarandache function and has been intensively studied [1]. Its behavior is quite erratic [2] and thus all we can reasonably hope for is a statistical approximation of its growth, e.g., an average. It appears that the sample mean \( E(S) \) satisfies [3]

\[
E(S(N)) = \frac{1}{N} \sum_{n=1}^{N} S(n) = O \left( \frac{N}{\ln(N)} \right)
\]

as \( N \) approaches infinity, but I don't know of a rigorous proof. A natural question is if some other sense of average might be more amenable to analysis.

Erdős [4,5] pointed out that \( P(n) = S(n) \) for almost all \( n \), meaning

\[
\lim_{N \to \infty} \frac{\left| \{n \leq N: P(n) < S(n)\} \right|}{N} = 0 \quad \text{that is,} \quad \left| \{n \leq N: P(n) < S(n)\} \right| = o(N)
\]

as \( N \) approaches infinity. Kastanas [5] proved this to be true, hence the following argument is valid. On one hand,

\[
\lambda = \lim_{n \to \infty} \frac{\ln(P(n))}{\ln(n)} \leq \lim_{n \to \infty} \frac{\ln(S(n))}{\ln(n)} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\ln(S(n))}{\ln(n)}
\]

The above summation, on the other hand, breaks into two parts:

\[
\lim_{N \to \infty} \frac{1}{N} \left( \sum_{P(n)=S(n)} \frac{\ln(P(n))}{\ln(n)} + \sum_{P(n)<S(n)} \frac{\ln(S(n))}{\ln(n)} \right)
\]

The second part vanishes:
\[
\lim_{N \to \infty} \frac{1}{N} \left\{ \sum_{P(n) = S(n)} \ln(S(n)) \right\} \leq \lim_{N \to \infty} \frac{1}{N} \left\{ \sum_{P(n) < S(n)} \ln(n) \right\} = \lim_{N \to \infty} \frac{o(N)}{N} = 0
\]

while the first part is bounded from above:

\[
\lim_{N \to \infty} \frac{1}{N} \left( \sum_{P(n) = S(n)} \frac{\ln(P(n))}{\ln(n)} \right) \leq \lim_{N \to \infty} \frac{1}{N} \cdot \sum_{n=1}^{N} \frac{\ln(P(n))}{\ln(n)} = \lim_{n \to \infty} E \left( \frac{\ln(P(n))}{\ln(n)} \right) = \lambda
\]

We deduce that

\[
\lim_{n \to \infty} E \left( \frac{\ln(S(n))}{\ln(n)} \right) = \lambda = 0.6243299885...
\]

where \( \lambda \) is the famous Golomb-Dickman constant [6-9]. Therefore \( \lambda \cdot n \) is the asymptotic average number of digits in the output of \( S \) at an \( n \)-digit input, that is, 62.43% of the original number of digits. As far as I know, this result about the Smarandache function has not been published before.

A closely related unsolved problem concerns estimating the variance of \( S \).

References