THE EQUATION \( a^2(k+2, S(n)) = a^2(k+1, S(n)) + a^2(k, S(n)) \)

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Abstract. For any positive integer \( a \), let \( S(a) \) be the Smarandache function of \( a \). For any positive integers \( r \) and \( b \), let \( a(r, b) \) be the first \( r \) digits of \( b \). In this paper we prove that the title equation has no positive integer solutions \( (n, k) \).

Key words: Smarandache function, diophantine equation

Let \( N \) be the set of all positive integers. For any positive integer \( a \), let \( S(a) \) be the Smarandache function of \( a \). For any positive integer \( b \) with \( s \) digits, let \( a(r, b) \) be the first \( r \) digits of \( b \). Recently, Beneze [1] proposed the following problem:

**Problem** Determine all solutions \( (n, k) \) of the equation

\( a^2(k + 2, S(n)) = a^2(k + 1, S(n)) + a^2(k, S(n)) \), \( n, k \in N \).

In this paper we completely solve the above-mentioned problem as follows.

**Theorem** The equation (3) has no solutions \( (n, k) \).

**Proof.** Let \( (n, k) \) be a solution of (3). It is a well known fact that \( S(n) \) is a positive integer (see [2]). Let \( b = S(n) \). We may assume that \( b \)
has \( s \) digits as (1). For any positive integer \( r \), by the definition (2) of \( a(r, b) \), we have

\[
\alpha(r + 1, b) = \begin{cases} 
10\alpha(r, b) + t_{s-r+1}, & \text{if } r < s, \\
\alpha(r, b), & \text{if } r \geq s.
\end{cases}
\]  

(4)

If \( k > s - 1 \), then from (4) we get \( a(k + 2, b) = a(k + 1, b) \). Hence, by (3), we obtain \( a(k, b) = 0 \), a contradiction.

If \( k < s - 1 \), then from (4) we get

\[
a(k + 2, b) \geq 10 \cdot a(k + 1, b).
\]

(5)

Hence, by (3) and (5), we get

\[
99 \cdot a^2(k + 1, b) \leq a^2(k, b).
\]

(6)

However, we see from (4) that \( a(k + 1, b) \geq a(k, b) \). Therefore, (6) is impossible. Thus, the equation (3) has no solutions \((n, k)\).

References


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