The general term of the prime number sequence and the Smarandache prime function.

Sebastián Martín Ruiz. Avda de Regla, 43 Chipiona 11550 Cádiz Spain.

Let's consider the function $d(i)$ = number of divisors of the positive integer number $i$. We have found the following expression for this function:

$$d(i) = \sum_{k=1}^{i} E\left(\frac{i}{k}\right) - E\left(\frac{i-1}{k}\right)$$

We proved this expression in the article “A functional recurrence to obtain the prime numbers using the Smarandache Prime Function”.

We deduce that the following function:

$$G(i) = -E\left[-\frac{d(i)-2}{2}\right]$$

This function is called the Smarandache Prime Function (Reference). It takes the next values:

$$G(i) = \begin{cases} 0 & \text{if } i \text{ is prime} \\ 1 & \text{if } i \text{ is compound} \end{cases}$$

Let is consider now $\pi(n)$ = number of prime numbers smaller or equal than $n$. It is simple to prove that:

$$\pi(n) = \sum_{i=2}^{n} (1 - G(i))$$

Let is have too:

If $1 \leq k \leq p_n - 1$ $\Rightarrow$ $E\left(\frac{m(k)}{n}\right) = 0$

If $C \geq k \geq p_n$ $\Rightarrow$ $E\left(\frac{m(k)}{n}\right) = 1$

We will see what conditions have to cany $C_n$.

Therefore we have the following expression for $p_n$ n-th prime number:

$$p_n = 1 + \sum_{k=1}^{C_n} (1 - E\left(\frac{m(k)}{\pi}\right))$$

If we obtain $C_n$ that only depends on $n$, this expression will be the general term of the prime numbers sequence, since $\pi$ is in function with $G$ and $G$ does with $d(i)$ that is expressed in function with $i$ too. Therefore the expression only depends on $n$.

$E[x]$=The highest integer equal or less than $n$
Let us consider \( C_n = 2(E(n \log n) + 1) \)

Since \( p_n \sim n \log n \) from of a certain \( n_0 \) it will be true that

\[
(1) \quad p_n \leq 2(E(n \log n) + 1)
\]

If \( n_0 \) it is not too big, we can prove that the inequality is true for smaller or equal values than \( n_0 \).

It is necessary to that:

\[
E\left[ \frac{m(C_n)}{n} \right] = 1
\]

If we check the inequality:

\[
(2) \quad \pi(2(E(n \log n) + 1)) < 2n
\]

We will obtain that:

\[
\frac{m(C_n)}{n} < 2 \Rightarrow E\left[ \frac{m(C_n)}{n} \right] \leq 1 \quad ; \quad C_n \geq p_n \Rightarrow E\left[ \frac{m(C_n)}{n} \right] = 1
\]

We can experimentaly check this last inequality saying that it checks for a lot of values and the difference tends to increase, which makes to think that it is true for all \( n \).

Therefore if we prove that the next inequalities are true:

\[
(1) \quad p_n \leq 2(E(n \log n) + 1), \\
(2) \quad \pi(2(E(n \log n) + 1)) < 2n
\]

which seems to be very probable; we will have that the general term of the prime numbers sequence is:

\[
p_n = 1 + \sum_{\omega=1}^{2(E(n \log n) + 1)} \left[ 1 - E\left( \sum_{\mu=1}^{k} \sum_{j=2}^{\left( \frac{x(\mu)}{1-(-1)^{\mu}n} - \epsilon \right)} j \right) \right]
\]

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If now we consider the general term defined in the same way but for all real number greater than zero the following graphic is obtained:

Let is observe that if $0 < x < 1 \ P(x) = 1$ si $x = 1 \ P(x) = 2$ and for $n - 1 < x \leq n \ P(x) = p_n$.

Reference:
Http://www.gallup.unm.edu/~Smarandache/primfnct.txt