Smarandache functions of first kind are defined in [1] thus:

$$S_n^*: N^* \rightarrow N^*, \quad S_n^*(k) = 1 \quad \text{and} \quad S_n^*(k) = \max_{1 \leq j \leq r} \{S_{p_j}(i_j, k)\},$$

where \( n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r} \) and \( S_{p_j} \) are functions defined in [4].

They \( \Sigma_1 \) standardise \( (N^*, +) \) in \( (N^*, \leq, +) \) in the sense that

$$\Sigma_1: \quad \max \{S_n(a), S_n(b)\} \leq S_n(a + b) \leq S_n(a) + S_n(b)$$

for every \( a, b \in N^* \) and \( \Sigma_2 \) standardise \( (N^*, +) \) in \( (N^*, \leq, \cdot) \) by

$$\Sigma_2: \quad \max \{S_n(a), S_n(b)\} \leq S_n(a + b) \leq S_n(a) \cdot S_n(b), \quad \text{for every} \quad a, b \in N^*$$

In [2] it is proved that the functions \( S_n^* \) are increasing and the sequence \( \{S_n^*(i)\}_{i \in N^*} \) is also increasing. It is also proved that if \( p, q \) are prime numbers, then

$$p \cdot i < q \Rightarrow S_{p^i} < S_q \quad \text{and} \quad i < q \Rightarrow S_i < S_q,$$

where \( i \in N^* \).

It would be used in this paper the formula

$$S_p(k) = p(k - i_k), \quad \text{for same} \ i_k \ \text{satisfying} \ 0 \leq i_k \leq \left[ \frac{k - 1}{p} \right], \quad \text{(see [3])} \quad (1)$$

1. Proposition. Let \( p \) be a prime number and \( k_1, k_2 \in N^* \). If \( k_1 < k_2 \) then \( i_{k_1} \leq i_{k_2} \), where \( i_{k_1}, i_{k_2} \) are defined by (1).

Proof. It is known that \( S_p^*: N^* \rightarrow N^* \) and \( S_p^*(k) = pk \) for \( k \leq p \). If \( S_p^*(k) = mp^a \) with \( m, a \in N^*, (m, p) = 1 \), there exist \( a \) consecutive numbers:

\( n, n + 1, \ldots, n + \alpha - 1 \) so that

\( k \in \{n, n + 1, \ldots, n + \alpha - 1\} \) and

\( S_p^*(n) = S_p^*(n + 1) = \cdots = S(n + \alpha - 1) \).
this means that $S_p$ is stationed the $\alpha - 1$ steps ($k \rightarrow k + 1$).

If $k_1 < k_2$ and $S_p(k_1) = S_p(k_2)$, because $S_p(k_1) = p(k_1 - ik_1)$, $S_p(k_2) = p(k_2 - ik_2)$ it results $i_{k_1} < i_{k_2}$.

If $k_1 < k_2$ and $S_p(k_1) < S_p(k_2)$, it is easy to see that we can write:

\[ i_{k_1} = \beta_1 + \sum_a (\alpha - 1) \]

where $\beta_1 = 0$ for $S_p(k_1) = mp^a$, if $S_p(k_1) = mp^a$

then $\beta_1 \in \{0,1,2,...,\alpha - 1\}$

and

\[ i_{k_1} = \beta_2 + \sum_a (\alpha - 1) \]

where $\beta_2 = 0$ for $S_p(k_2) = mp^a$, if $S_p(k_2) = mp^a$ then

\[ mp^a < S_p(k_1) \]

$\beta_2 \in \{0,1,2,...,\alpha - 1\}$.

Now is obviously that $k_1 < k_2$ and $S_p(k_1) < S_p(k_2) \Rightarrow i_{k_1} \leq i_{k_2}$. We note that, for $k_1 < k_2$, $i_{k_1} = i_{k_2}$ iff $S_p(k_1) < S_p(k_2)$ and \( \{mp^a | \alpha > 1 \text{ and } mp^a \leq S_p(k_1)\} = \{mp^a | \alpha > 1 \text{ and } mp^a < S_p(k_2)\} \)

2. Proposition. If $p$ is a prime number and $p \geq 5$, then $S_p > S_{p-1}$ and $S_p > S_{p+1}$.

Proof. Because $p - 1 < p$ it results that $S_{p-1} < S_p$. Of course $p + 1$ is even and so:

(i) if $p + 1 = 2l$, then $i > 2$ and because $2i < 2l - 1 = p$ we have $S_{p+1} < S_p$.

(ii) if $p + 1 = 2l$, let $p + 1 = p^l_1 \cdot p^l_2 \cdot ... \cdot p^l_r$, then $S_{p+1}(k) = \max_{i \in \mathbb{N}} \{S_{p+1}(k)\} = S_{p+1}(k) = S_p(i^m \cdot k)$.

Because $p^m \cdot i^m \leq \frac{p+1}{2} < p$ it results that $S_{p^m}(k) < S_p(k)$ for $k \in \mathbb{N}^*$, so that $S_{p+1} < S_p$.

3. Proposition. Let $p,q$ be prime numbers and the sequences of functions

\[ \{S_p\}_{i \in \mathbb{N}^*}, \{S_q\}_{j \in \mathbb{N}^*} \]

If $p < q$ and $i \leq j$, then $S_p(i) < S_q(j)$.

Proof. Evidently, if $p < q$ and $i \leq j$, then for every $k \in \mathbb{N}^*$

\[ S_p(k) \leq S_q(k) \]

so,

\[ S_p(k) < S_q(k) \]

4. Definition. Let $p,q$ be prime numbers. We consider a function $S_{q,j}$ a sequence of functions $\{S_p\}_{i \in \mathbb{N}^*}$, and we note:

\[ i_{(j)} = \max \left\{ i | S_p(i) < S_q(j) \right\} \]
\[ i^{(j)} = \min \{ i | S_q^i < S_q^{i(j)} \}. \]

then \( \{ k \in \mathbb{N} | i_{(j)} < k < i^{(j)} \} = \Delta_{i \rightarrow (q')} = \Delta_{i(j)} \) defines the interference zone of the function \( S_q^i \) with the sequence \( \{ S_q^i \}_{i \in \mathbb{N}} \).

5. Remarque.
   a) If \( S_q^i < S_p^j \) for \( i \in \mathbb{N}^* \), then now exists \( i(j) \) and \( j \not\in \mathbb{N} \), and we say that \( S_q^i \) is separately of the sequence of functions \( \{ S_q^i \}_{i \in \mathbb{N}} \).
   b) If there exist \( k \in \mathbb{N}^* \) so that \( S_q^i < S_q^j < S_p^k \), then \( \Delta_{i \rightarrow (q')} = \emptyset \) and say that the function \( S_q^i \) does not interfere with the sequence of functions \( \{ S_q^i \}_{i \in \mathbb{N}} \).

6. Definition. The sequence \( \{ x_n \}_{n \in \mathbb{N}} \) is generally increasing if

\[ \forall n \in \mathbb{N}^* \exists n_0 \in \mathbb{N}^* \text{ so that } x_m \geq x_n \text{ for } m \geq m_0. \]

7. Remarque. If the sequence \( \{ x_n \}_{n \in \mathbb{N}} \) with \( x_n \geq 0 \) is generally increasing and bounded, then every subsequence is generally increasing and bounded.

8. Proposition. The sequence \( \{ S_n(k) \}_{n \in \mathbb{N}} \), where \( k \in \mathbb{N}^* \), is in generally increasing and bounded.

   Proof. Because \( S_n(k) = S_n^*(k) \), it results that \( \{ S_n(k) \}_{n \in \mathbb{N}} \) is a subsequence of \( \{ S_m(1) \}_{m \in \mathbb{N}} \).
   The sequence \( \{ S_m(1) \}_{m \in \mathbb{N}} \) is generally increasing and bounded because:

\[ \forall m \in \mathbb{N}^* \exists t_0 = m! \text{ so that } \forall t \geq t_0 S_n(t) \geq S_n^*(1) = m \geq S_m(1). \]

From the remarque 7 it results that the sequence \( \{ S_n(k) \}_{n \in \mathbb{N}} \) is generally increasing and bounded.

9. Proposition. The sequence of functions \( \{ S_n \}_{n \in \mathbb{N}} \) is generally increasing bounded.

   Proof. Obviously, the zone of interference of the function \( S_m \) with \( \{ S_n \}_{n \in \mathbb{N}} \) is the set

\[ \Delta_{m(n)} = \{ k \in \mathbb{N}^* | n_{(m)} < k < n^{(m)} \} \text{ where} \]

\[ n_{(m)} = \max \{ n \in \mathbb{N}^* | S_n < S_m \} \]

\[ n^{(m)} = \min \{ n \in \mathbb{N}^* | S_m < S_n \} \]
The interference zone $\Delta_{m(m)}$ is nonempty because $S_m \in \Delta_{m(m)}$ and finite for $S_1 \leq S_m \leq S_p$, where $p$ is one prime number greater than $m$.

Because $\{S_n(1)\}$ is generally increasing it results:

$$\forall m \in \mathbb{N^*} \quad \exists t_0 \in \mathbb{N^*} \quad \text{so that} \quad S_{t_0}(1) \geq S_m(1) \text{ for } \forall t \geq t_0.$$  

For $r_0 = t_0 + n^{(m)}$ we have

$$S_r \geq S_m \geq S_m(1) \text{ for } \forall r \geq r_0,$$

so that $\{S_r\}_{n \in \mathbb{N^*}}$ is generally increasing boundless.

10. Remarque.

a) For $n = p_1^{(1)} \cdot p_2^{(2)} \cdots p_r^{(r)}$ are possible the following cases:

1) $\exists k \in \{1,2,\ldots,r\}$ so that

$$S_{p_k^{(k)}} \leq S_{p_{j_k}^{(j_k)}} \text{ for } j \in \{1,2,\ldots,r\},$$

then $S_n = S_{p_k^{(k)}}$ and $p_k^{(k)}$ is named the dominant factor for $n$.

2) $\exists k_1,k_2,\ldots,k_m \in \{1,2,\ldots,r\}$ so that:

$$\forall t \in \overline{1,m} \quad \exists q_t \in \mathbb{N^*} \quad \text{so that} \quad S_n(q_t) = S_{p_{k_t}^{(k_t)}}(q_t) \text{ and}$$

$$\forall I \in \mathbb{N^*} \quad S_n(I) = \max_{1 \leq t \leq m} \left\{ S_{p_{k_t}^{(k_t)}}(I) \right\}.$$  

We shall name $\{p_{k_t}^{(k_t)} \mid t \in \overline{1,m}\}$ the active factors, the others would be named passive factors for $n$.

b) We consider

$$N_{p_1,p_2} = \{n = p_1^{(1)} \cdot p_2^{(2)} \mid 1,2 \in \mathbb{N^*} \}, \text{ where } p_1 < p_2 \text{ are prime numbers}.$$  

For $n \in N_{p_1,p_2}$ appear the following situations:

1) $i_1 \in (0,i_1^{(1)})$, this means that $p_1^{(1)}$ is a passive factor and $p_2^{(2)}$ is an active factor.

2) $i_1 \in (i_1^{(1)},i_1^{(2)})$ this means that $p_1^{(1)}$ and $p_2^{(2)}$ are active factors.

3) $i_1 \in [i_1^{(2)},\infty)$ this means that $p_1^{(1)}$ is an active factor and $p_2^{(2)}$ is a passive factor.
For $p_1 < p_2$ the repartition of exponents is represented in the following scheme:

For numbers of type 2) $i_1 \in (i_{i(2)}, i_{2(i)})$ and $i_2 \in (i_{2(i)}, i_{3(i)})$

c) I consider that

$$N_{p_1 p_2 p_3} = \{ n = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} | i_1, i_2, i_3 \in N^* \},$$

where $p_1 < p_2 < p_3$ are prime numbers.

Exist the following situations:

1) $n \in N^{p_j}, j = 1, 2, 3$ this means that $p_j^{i_j}$ is active factor.

2) $n \in N^{p_j p_k}, j = k; j, k \in \{1, 2, 3\}$, this means that $p_j^{i_j}, p_k^{i_k}$ are active factors.

3) $n \in N^{p_j p_k p_3},$ this means that $p_j^{i_j}, p_k^{i_k}, p_3^{i_3}$ are active factors. $N^{p_j p_k p_3}$ is named the $S$-active cone for $N_{p_1 p_2 p_3}$.

Obviously

$$N^{p_j p_k p_3} = \{ n = p_1^{i_1} p_2^{i_2} p_3^{i_3} | i_1, i_2, i_3 \in N^* \text{ and } i_k \in (i_{k(i)}, i_{k(d)}) \text{ where } j = k; j, k \in \{1, 2, 3\} \}.$$
For \( p_1 < p_2 \) the repartition of exponents is respectively in following scheme:

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\[ p_1 < p_2 \]
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For numbers of type 2) \( i_1 \in \{i_{(u_2)}, i_{(u_2)}^{(u_3)}\} \) and \( i_2 \in \{i_{(u_4)}, i_{(u_4)}^{(u_3)}\} \)

c) I consider that

\[
N_{p_1 p_2 p_3} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} | i_1, i_2, i_3 \in \mathbb{N}^*\},
\]

where \( p_1 < p_2 < p_3 \) are prime numbers.

Exist the following situations:

1) \( n \in N^{p_j}, j = 1, 2, 3 \) this means that \( p_j^{i_j} \) is active factor.

2) \( n \in N^{p_j p_k}, j = k; j, k \in \{1, 2, 3\} \), this means that \( p_j^{i_j}, p_k^{i_k} \) are active factors.

3) \( n \in N^{p_1 p_2 p_3} \), this means that \( p_1^{i_1}, p_2^{i_2}, p_3^{i_3} \) are active factors. \( N^{p_1 p_2 p_3} \) is named the S-active cone for \( N_{p_1 p_2 p_3} \).

Obviously

\[
N^{p_1 p_2 p_3} = \{n = p_1^{i_1} p_2^{i_2} p_3^{i_3} | i_1, i_2, i_3 \in \mathbb{N}^* \ and \ i_k \in \{i_{(u_4)}, \ell_{(u_4)}^{(u_3)}\} \ where \ j \neq k; j, k \in \{1, 2, 3\}\}.
\]

The repartition of exponents is represented in the following scheme:
d) Generally, I consider $N_{p_1 p_2 \cdots p_r} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r} | i_1, i_2, \ldots, i_r \in \mathbb{N}^*\}$, where $p_1 < p_2 < \cdots < p_r$ are prime numbers.

On $N_{p_1 p_2 \cdots p_r}$ exist the following relation of equivalence:

\[ n \rho m \iff n \text{ and } m \text{ have the same active factors.} \]

This have the following classes:

- $N_{p_n}$, where $j_1 \in \{1,2,\ldots,r\}$.

$n \in N_{p_n} \iff n$ has only $p_n^{i_{j_1}}$ active factor

- $N_{p_n p_{j_2}}$, where $j_1 \neq j_2$ and $j_1, j_2 \in \{1,2,\ldots,r\}$.

$n \in N_{p_n p_{j_2}} \iff n$ has only $p_n^{i_{j_1}}, p_{j_2}^{i_{j_2}}$ active factors.

- $N_{p_1 p_2 \cdots p_r}$ which is named S-active cone.

$N_{p_1 p_2 \cdots p_r} = \{n \in N_{p_1 p_2 \cdots p_r} | n \text{ has } p_1^{i_1}, p_2^{i_2}, \ldots, p_r^{i_r} \text{ active factors}\}$.

Obviously, if $n \in N_{p_1 p_2 \cdots p_r}$, then $i_k = (i_{k(j_1)}, i_{k(j_2)})$ with $k = j$ and $j, k \in \{1,2,\ldots,r\}$.

REFERENCES


