The sequence of prime numbers

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This article lets out a law of recurrence in order to obtain the sequence of prime numbers \( \{p_k\}_{k \geq 1} \) expressing \( p_{k+1} \) as a function of \( p_1, p_2, \cdots, p_k \).

Suppose we can find a function \( G_k(n) \) with the following property:

\[
G_k(n) = \begin{cases} 
-1 & \text{if } n < p_{k+1} \\
0 & \text{if } n = p_{k+1} \\
\text{something} & \text{if } n > p_{k+1}
\end{cases}
\]

This is a variation of the Smarandache Prime Function [2].

Then we can write down a recurrence formula for \( p_k \) as follows.

Consider the product:

\[
\prod_{s=p_k+1}^{m} G_k(s)
\]

If \( p_k < m < p_{k+1} \) one has

\[
\prod_{s=p_k+1}^{m} G_k(s) = \prod_{s=p_k+1}^{m} (-1) = (-1)^{m-p_k}
\]

If \( m \geq p_{k+1} \)

\[
\prod_{s=p_k+1}^{m} G_k(s) = 0
\]

since \( G_k(p_{k+1}) = 0 \)

Hence

\[
\sum_{m=p_k+1}^{p_{k+1}-1} (-1)^{m-p_k} \prod_{s=p_k+1}^{m} G_k(s) = \\
\sum_{m=p_k+1}^{p_{k+1}-1} (-1)^{m-p_k} \prod_{s=p_k+1}^{m} G_k(s) + \sum_{m=p_k+1}^{2p_k} (-1)^{m-p_k} \prod_{s=p_k+1}^{m} G_k(s)
\]

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The second addition is zero since all the products we have the factor $G_k(p_{k+1}) = 0$

\[ = \sum_{m=p_k+1}^{p_{k+1}-1} (-1)^{m-p_k} (-1)^{m-p_k} \]

\[ = p_{k+1} - 1 - (p_k + 1) + 1 = p_{k+1} - p_k - 1 \]

so

\[ p_{k+1} = p_k + 1 + \sum_{m=p_k+1}^{2p_k} (-1)^{m-p_k} \prod_{i=p_k+1}^{m} G_i(s) \]

which is a recurrence relation for $p_k$.

We now show how to find such a function $G_k(n)$ whose definition depends only on the first $k$ primes and not on an explicit knowledge of $p_{k+1}$.

And to do so we define\(^1\):

\[ T_k(n) = \sum_{i_1=0}^{\log_{p_1} n} \sum_{i_2=0}^{\log_{p_2} n} \cdots \sum_{i_k=0}^{\log_{p_k} n} \left( \prod_{s=1}^{k} \binom{n}{p_s^i} \right) \]

Let's see the value which $T_k(n)$ takes for all $n \geq 2$ integer. We distinguish two cases:

Case 1: $n < p_{k+1}$

The expression $p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}$ with $i_1 = 0, 1, 2 \cdots \log_{p_1} n$, $i_2 = 0, 1, 2 \cdots \log_{p_2} n$

... $i_k = 0, 1, 2 \cdots \log_{p_k} n$ all the values occur 1, 2, 3, ..., $n$ each one of them only once and moreover some more values, strictly greater than $n$.

We can look at it. If $1 \leq m \leq n$ one obtains that $m < p_{k+1}$ for which $1 \leq m = p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k} \leq n$. From where one deduces that $1 \leq p_{k+1}^n \leq n$ and for it $0 \leq \alpha_s \leq \log_{p_s} n$ for all $s = 1, \ldots, k$

Therefore, for $i_s = \alpha_s$, $s = 1, 2, \ldots, k$ we have the value $m$. This value only appears once, the prime number decomposition of $m$ is unique.

In fact the sums of $T_k(n)$ can be achieved up to the highest power of $p_k$ contained in $n$ instead of $\log_{p_k} n$.

Therefore one has that

\[ T_k(n) = \sum_{i_1=0}^{\log_{p_1} n} \sum_{i_2=0}^{\log_{p_2} n} \cdots \sum_{i_k=0}^{\log_{p_k} n} \left( \prod_{s=1}^{k} \binom{n}{p_s^i} \right) = \left( \binom{n}{1} \right) + \left( \binom{n}{2} \right) + \cdots + \left( \binom{n}{n} \right) = 2^n - 1 \]

\(^1\)Given that $i_s$, $s = 1, 2, \ldots, k$ only takes integer values one appreciates that the sums of $T_k(n)$ are until $E(\log_{p_s} n)$ where $E(x)$ is the greatest integer less than or equal to $x$. 

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since, in the case $p_1 p_2 \cdots p_k$ would be greater than $n$ one has that:

$$\left( \prod_{s=1}^{k} p_s^{i_s} \right)^n = 0$$

Case 2: $n = p_{k+1}$

The expression $p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}$ with $i_1 = 0, 1, 2 \cdots \log p_1$, $i_2 = 0, 1, 2 \cdots \log p_2$, $n$ \ldots $i_k = 0, 1, 2 \cdots \log p_k$, $n$ the values occur 1, 2, 3, \ldots, $p_{k+1} - 1$ each one of them only once and moreover some more values, strictly greater than $p_{k+1}$. One demonstrates in a form similar to case 1. It doesn't take the value $p_{k+1}$ since it is coprime with $p_1, p_2, \ldots, p_k$.

Therefore,

$$T_k(n) = \left( \frac{n}{1} \right) + \left( \frac{n}{2} \right) + \cdots + \left( \frac{n}{n-1} \right) = 2^n - 2$$

In case 3: $n > p_{k+1}$ it is not necessary to consider it.

Therefore, one has:

$$T_k(n) = \begin{cases} 2^n - 1 & \text{if } n < p_{k+1} \\ 2^n - 2 & \text{if } n = p_{k+1} \\ \text{something} & \text{if } n > p_{k+1} \end{cases}$$

and as a result:

$$G_k(n) = 2^n - 2 - T_k(n)$$

This is the summarized relation of recurrence:

Let's take $p_1 = 2$ and for $k \geq 1$ we define:

$$T_k(n) = \sum_{i_1=0}^{\log p_1 n} \sum_{i_2=0}^{\log p_2 n} \cdots \sum_{i_k=0}^{\log p_k n} \left( \prod_{s=1}^{k} p_s^{i_s} \right)^n$$

$$G_k(n) = 2^n - 2 - T_k(n)$$

$$p_{k+1} = p_k + 1 + \sum_{n=p_k+1}^{2p_k} (-1)^{n-p_k} \prod_{s=p_k+1}^{m} G_k(s)$$
References:


(2) E. Burton, "Smarandache Prime and Coprime Functions",
http://www.gallup.unm.edu/~smarandache/primfnct.txt

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