THE SMARANDACHE \( \varphi \)-SEQUENCE

Maohua Le
Department of Mathematics
Zhanjiang Normal College
29 Cunjin Road, Chikan
Zhanjiang, Guangdong
P.R.China

Abstract: In this paper we completely determine the Smarandache \( \varphi \)-sequence.

Key words: Smarandache \( \varphi \)-sequence; Euler totient function; diophantine equation

For any positive integer \( n \), let \( \varphi(n) \) be the Euler totient function of \( n \). Further, let the set

\[ A = \{ n | n = k \varphi(n) \text{, where } k \text{ is a positive integer} \}. \] (1)

Then, all elements \( n \) of \( A \) form the Smarandache \( \varphi \)-sequence (see [2]). In this paper we completely determine this sequence as follows.

Theorem. Let \( \{ a(x) \}_{x=1}^{\infty} \) be the Smarandache \( \varphi \)-sequence. Then we have

\[ a(x)=\begin{cases} 1, & \text{if } x = 1, \\ 2, & \text{if } x = 2, \\ 2^{(x+1)/2}, & \text{if } x>1 \text{ and } x \text{ is odd}, \\ 2^{x/2-1.3}, & \text{if } x>1 \text{ and } x \text{ is even}. \end{cases} \]

(2)

Supported by the National Natural Science Foundation of China (No.10271104), the Guangdong Provincial Natural Science Foundation (No.011781) and the Natural Science Foundation of the Education Department of Guangdong Province (No.0161).
Proof. We first consider the elements of $A$. We see from (1) that these elements are solutions of the equation

$$n = k \varphi(n).$$

(3)

Clearly, $(n,k)=(1,1)$ is a positive integer of (3). If $n \geq 1$, let

$$n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$$

be the factorization of $n$. By [1, Theorem 62], we have

$$\varphi(n) = p_1^{a_1-1} p_2^{a_2-1} \cdots p_s^{a_s-1} (p_1-1)(p_2-1) \cdots (p_s-1).$$

(5)

Substitute (4) and (5) into (3), we get

$$p_1 p_2 \cdots p_s = k(p_1-1)(p_2-1) \cdots (p_s-1).$$

(6)

If $n$ is even, then $p_1=2$ and $p_2, \cdots, p_s$ are odd primes. Since $p_i-1$ ($i=2, \cdots, s$) are even integers, we find from (6) that either $s=1$ and $k=2$ or $s=2$, $p_2=3$ and $k=3$. It follows that (3) has positive integer solutions $(n,k)=(2r,2)$ and $(2^r,3,3)$, where $r$ is a positive integer.

If $n$ is odd, then (6) is impossible, since $p_j$ ($j=1,2,\cdots,s$) are odd primes and $p_j-1$ ($j=1,2,\cdots,s$) are even integers.

Thus, by the above analysis, we obtain (2) immediately.

References
