THE SOLUTION OF THE DIOPHANTINE EQUATION $\sigma_n(n) = n$ ($\Omega$)

by Pál Grónás

This problem is closely connected to Problem 29916 in the first issue of the "Smarandache Function Journal" (see page 47 in [1]). The question is: "Are there an infinity of nonprimes $n$ such that $\sigma_n(n) = n$?". My calculations will show that the answer is negative.

Let us move on to the first step in deriving the solution of ($\Omega$). As the wording of Problem 29916 indicates, ($\Omega$) is satisfied if $n$ is a prime. This is not the case for $n = 1$ because $\sigma_n(1) = 0$.

Suppose $\prod_{i=1}^{k} p_i^{r_i}$ is the prime factorization of a composite number $n \geq 4$, where $p_1, \ldots, p_k$ are distinct primes, $r_i \in \mathbb{N}$ and $p_i \geq r_i$ for all $i \in \{1, \ldots, k\}$ and $p_i < p_{i+1}$ for all $i \in \{2, \ldots, k-1\}$ whenever $k \geq 3$.

First of all we consider the case where $k = 1$ and $r_1 \geq 2$. Using the fact that $\eta(p_i^{r_i}) = p_i s_i$ we see that

$\sigma_n(n) = \sigma_n(p_i^{r_i}) = \sum_{d_{i=0}}^{r_i} \eta(p_i^{r_i}) = \sum_{d=0}^{\max{p_i, r_i+1}} \eta(p_i^{r_i+1})$.

Therefore $2 p_i^{r_i-1} \leq r_1 (r_1 + 1)$ ($\Omega_1$) for some $r_1 \geq 2$. For $p_1 \geq 5$ this inequality ($\Omega_1$) is not satisfied for any $r_1 \geq 2$. So $p_1 < 5$, which means that $p_1 \in \{2, 3\}$. By the help of ($\Omega_1$) we can find a supremum for $r_1$ depending on the value of $p_1$. For $p_1 = 2$ the actual candidates for $r_1$ are 2, 3, 4 and for $p_1 = 3$ the only possible choice is $r_1 = 2$. Hence there are maximum 4 possible solution of ($\Omega$) in this case, namely $n = 4, 8, 9$ and 16. Calculating $\sigma_n(n)$ for each of these 4 values, we get $\sigma_n(4) = 6$, $\sigma_n(8) = 10$, $\sigma_n(9) = 9$ and $\sigma_n(16) = 16$. Consequently the only solutions of ($\Omega$) are $n = 9$ and $n = 16$.

Next we look at the case when $k \geq 2$:

$$n = \sigma_n(n)$$

Substituting $n$ with it's prime factorization we get

$$\prod_{i=1}^{k} p_i^{r_i} = \sigma_n(\prod_{i=1}^{k} p_i^{r_i}) = \sum_{d_{i=0}}^{r_i} \eta(d) = \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \eta(\prod_{i=1}^{k} p_i^{s_i})$$

$$= \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \max\{\eta(p_1^{s_1}), \ldots, \eta(p_k^{s_k})\}$$

$$\leq \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \max\{p_1 s_1, \ldots, p_k s_k\} \text{ since } \eta(p_i^{s_i}) \leq p_i s_i$$

$$< \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \max\{p_1 r_1, \ldots, p_k r_k\} \text{ because } s_i \leq r_i$$

$$= \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} p_1 r_1 \cdots p_k r_k \text{ (} p_1 r_1 \geq p_i r_i \text{ for } i \geq 2$$

$$\leq P_1 r_1 \prod_{i=1}^{k} (r_i + 1).$$
which is equivalent to
\[
\prod_{i=2}^{k} \frac{p_i r_i}{r_i + 1} < \frac{p_i r_i (r_i + 1)}{p_i^2} = \frac{r_i (r_i + 1)}{p_i^2 - 1} \quad (\Omega_2)
\]

This inequality motivates a closer study of the functions \( f(x) = \frac{a^x}{x+1} \) and \( g(x) = \frac{2^{(x-1)}}{x+1} \) for \( x \in [1, \infty) \), where \( a \) and \( b \) are real constants \( \geq 2 \). The derivatives of these two functions are
\[
\begin{align*}
    f'(x) &= \frac{a^x \ln a}{(x+1)^2} \quad \text{and} \quad \frac{x}{x+1} \ln a - 1 \geq (1 + 1) \ln 2 - 1 = 2 \ln 2 - 1 > 0.
\end{align*}
\]

So \( f(x) \) is increasing on \([1, \infty)\). Moreover \( g(x) \) reaches its absolute maximum value for \( x = \max\{1, \frac{2-\ln 5 + \sqrt{(\ln 5)^2 - 4}}{2 \ln 5} \} \).

Now \( \sqrt{(\ln b)^2 + 4} < \ln b + 2 \) for \( b \geq 2 \), which implies that \( x < \frac{(2-\ln 5) + (\ln 5 + 2)}{2 \ln 5} = \frac{2}{\ln 5} \leq \frac{2}{\ln 2} < 3 \). Furthermore it is worth mentioning that \( f(x) \to \infty \) and \( g(x) \to 0 \) as \( x \to \infty \).

Applying this to our situation means that \( \frac{p_i}{r_i + 1} (i \geq 2) \) is strictly increasing from \( \frac{b}{2} \) to \( \infty \). Besides \( \frac{r_i (r_i + 1)}{p_i} \leq \max\{2, \frac{4}{p_i}, \frac{12}{p_i^2} \} = \max\{2, \frac{4}{p_i} \} \leq 3 \) because \( \frac{4}{p_i} \geq \frac{12}{p_i^2} \) whenever \( p_i \geq 2 \).

Combining this knowledge with \((\Omega_2)\) we get that \( \prod_{i=2}^{k} \frac{p_i}{r_i + 1} < \prod_{i=2}^{k} \frac{r_i (r_i + 1)}{p_i} \leq \prod_{i=2}^{k} \frac{r_i (r_i + 1)}{p_i} \leq 3 (\Omega_3) \) for all \( r_i \in \mathbb{N} \). In other words, \( \prod_{i=2}^{k} \frac{p_i}{r_i + 1} < 3 \). Consequently \( \prod_{i=2}^{k} \frac{p_i}{r_i + 1} < 3 \) implies that \( k \leq 3 \).

Let us assume \( k = 2 \). Then \((\Omega_2)\) and \((\Omega_3)\) state that \( \frac{p_2}{r_2 + 1} < \frac{r_2 (r_2 + 1)}{p_2} \) and \( \frac{p_2}{r_2 + 1} < 3 \), i.e. \( p_2 < 6 \). Next we suppose \( r_2 \geq 3 \). It is obvious that \( p_1 p_2 \geq 2 \cdot 3 = 6 \), which is equivalent to \( p_2 \geq \frac{6}{p_1} \). Using this fact we get \( \frac{p_2}{r_2 + 1} < \frac{r_2 (r_2 + 1)}{p_2} \leq \max\{2, \frac{4}{p_1} \} \leq \max\{2, p_2 \} = p_2 \), so \( p_2 < 4 \). Accordingly \( p_2 < 2 \), a contradiction which implies that \( r_2 \leq 2 \). Hence \( p_2 \in \{2, 3, 5\} \) and \( r_2 \in \{1, 2\} \).

Furthermore \( 1 \leq \frac{p_2}{r_2 + 1} < \frac{r_2 (r_2 + 1)}{p_2} \leq \frac{r_2 (r_2 + 1)}{\frac{2 r_2}{r_2 + 1}} \), which implies that \( r_1 \leq 6 \). Consequently, by fixing the values of \( p_2 \) and \( r_2 \), the inequalities \( \frac{r_1 (r_1 + 1)}{p_1} > \frac{p_2}{r_2 + 1} \) and \( p_1 r_1 \geq p_2 r_2 \) give us enough information to determine a supremum (less than 7) for \( r_1 \) for each value of \( p_1 \).

This is just what we have done, and the result is as follows:

<table>
<thead>
<tr>
<th>( p_2 )</th>
<th>( r_2 )</th>
<th>( p_1 )</th>
<th>( r_1 )</th>
<th>( n = p_1^p p_2^q )</th>
<th>( \sigma_n(n) )</th>
<th>IF ( \sigma_n(n) = n ) THEN</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>2 \cdot 3^1</td>
<td>2 \cdot 3 r_1 (r_1 + 1)</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>5</td>
<td>1</td>
<td>2 \cdot 5^1</td>
<td>2 \cdot 5 r_1 (r_1 + 1)</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>( p_1 \geq 7 )</td>
<td>1</td>
<td>2 p_1</td>
<td>2 + 2 p_1</td>
<td>0 = 2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>2 \cdot 7^1</td>
<td>34</td>
<td>34 = 36</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>( p_1 \geq 5 )</td>
<td>1</td>
<td>3 p_1</td>
<td>3 p_1 + 6</td>
<td>( p_1 = 6 )</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1 \leq r_1 \leq 5</td>
<td>3 \cdot 2^1</td>
<td>3 r_1^2 - 2 r_1 + 12</td>
<td>r_1 = 3</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>( p_1 \geq 5 )</td>
<td>1</td>
<td>3 p_1</td>
<td>2 p_1 + 3</td>
<td>( p_1 = 3 )</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>40</td>
<td>30</td>
<td>30 = 40</td>
</tr>
</tbody>
</table>

By looking at the rightmost column in the table above, we see that there are only contradictions except in the case where \( n = 3 \cdot 2^3 \) and \( r_1 = 3 \). So \( n = 3 \cdot 2^3 = 24 \) and \( \sigma_n(24) = 24 \).

In other words, \( n = 24 \) is the only solution of \((\Omega)\) when \( k = 2 \).
Finally, suppose $k = 3$. Then we know that $\frac{p_1}{2} \cdot \frac{p_2}{2} < 3$, i.e. $p_2 p_3 < 12$. Hence $p_2 = 2$ and $p_3 \geq 3$. Therefore $\frac{r_3}{p_1^{r_3}} = \frac{r_3}{p_1^{r_3}} < 2$ \((\Omega_4)\) and by applying \((\Omega_3)\) we find that 
\[ \prod_{i=1}^{3} \frac{p_i}{i} = \frac{p_1}{2} < 2, \text{ giving } p_3 = 3. \]

Combining the two inequalities \((\Omega_3)\) and \((\Omega_4)\) we get that $\frac{p_1^2}{r_2 + 1} \cdot \frac{p_3^2}{r_3 + 1} < 2$. Knowing that the left side of this inequality is a product of two strictly increasing functions on \([1, \infty)\), we see that the only possible choices for $r_2$ and $r_3$ are $r_2 = r_3 = 1$. Inserting these values in \((\Omega_3)\), we get $\frac{p_1}{1 + 1} \cdot \frac{p_3}{1 + 1} = \frac{3}{3} < \frac{r_3(r_3+1)}{p_1^{r_3+1}} \leq \frac{r_3(r_3+1)}{p_3^{r_3+1}}$. This implies that $r_1 = 1$. Accordingly \((\Omega)\) is satisfied only if $n = 2 \cdot 3 \cdot p_1 = 6 p_1$:

\[
6 p_1 = \sigma_\eta(6 p_1) = \eta(1) + \eta(2) + \eta(3) + \eta(6) + \sum_{i=0}^{1} \sum_{j=0}^{1} \eta(2^i 3^j p_1) \\
= 0 + 2 + 3 + 3 + \sum_{i=0}^{1} \sum_{j=0}^{1} \max\{ \eta(p_1), \eta(2^i 3^j) \} \\
= 8 + \sum_{i=0}^{1} \sum_{j=0}^{1} \max\{ p_1, \eta(2^i 3^j) \} \\
= 8 + 4 p_1 \text{ because } \eta(2^i 3^j) \leq 3 < p_1 \text{ for all } i, j \in \{0,1\} \\
\]

which contradicts the fact that $p_1 \geq 5$. Therefore \((\Omega)\) has no solution for $k = 3$.

**Conclusion**: $\sigma_\eta(n) = n$ if and only if $n$ is a prime, $n = 9$, $n = 16$ or $n = 24$.

**Remark**: A consequence of this work is the solution of the inequality $\sigma_\eta(n) > n \ (**\)$. This solution is based on the fact that \((\ast)\) implies \((\Omega_2)\).

So $\sigma_\eta(n) > n$ if and only if $n = 8, 12, 18, 20$ or $n = 2p$ where $p$ is a prime. Hence $\sigma_\eta(n) \leq n + 4$ for all $n \in \mathbb{N}$.

Moreover, since we have solved the inequality $\sigma_\eta(n) \geq n$, we also have the solution of $\sigma_\eta(n) < n$.

**References**


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