THE THIRD AND FOURTH CONSTANTS OF SMARANDACHE

by

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In the present note we prove the divergence of some series involving the Smarandache function, using an unitary method, and then we prove that the series

\[ \sum_{n=2}^{\infty} \frac{1}{S(2)S(3) \ldots S(n)} \]

is convergent to a number \( s \in (71/100, 101/100) \) and we study some applications of this series in the Number Theory (third constant of Smarandache).

The Smarandache Function \( S : \mathbb{N}^* \rightarrow \mathbb{N} \) is defined [1] such that \( S(n) \) is the smallest integer \( k \) with the property that \( k! \) is divisible by \( n \).

Proposition 1. If \((x_n)_{n \geq 1}\) is a strict increasing sequence of natural numbers, then the series:

\[ \sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}, \quad (1) \]

where \( S \) is the Smarandache function, is divergent.

Proof. We consider the function \( f : [x_n, x_{n+1}] \rightarrow \mathbb{R} \), defined by \( f(x) = \ln \ln x \). It fulfils the conditions of the Lagrange's theorem of finite increases. Therefore there is \( c_n \in (x_n, x_{n+1}) \) such that:

\[ \ln \ln x_{n+1} - \ln \ln x_n = \frac{1}{c_n \ln c_n} (x_{n+1} - x_n). \quad (2) \]

Because \( x_n < c_n < x_{n+1} \), we have:

\[ \frac{x_{n+1} - x_n}{x_{n+1} \ln x_{n+1}} < \ln \ln x_{n+1} - \ln \ln x_n < \frac{x_{n+1} - x_n}{x_n \ln x_n}, \quad (\forall) n \in \mathbb{N}, \quad (3) \]

if \( x_n = 1 \).
We know that for each \( n \in \mathbb{N}^* \setminus \{1\} \), \( \frac{S(n)}{n} \leq 1 \), i.e.

\[
0 < \frac{S(n)}{n \ln n} \leq \frac{1}{\ln n}.
\] (4)

from where it results that \( \lim_{n \to \infty} \frac{S(n)}{n \ln n} = 0 \). Hence there exists \( k > 0 \) such that

\[
\frac{S(n)}{n \ln n} < k, \text{ i.e., } n \ln n > \frac{S(n)}{k} \text{ for any } n \in \mathbb{N}^*,
\]

so

\[
\frac{1}{x_n \ln x_n} < \frac{k}{S(x_n)}.
\] (5)

Introducing (5) in (3) we obtain:

\[
\ln \ln x_{n+1} - \ln \ln x_n < k \frac{x_{n+1} - x_n}{S(x_n)}, \quad (\forall) n \in \mathbb{N}^* \setminus \{1\}.
\] (6)

Summing up after \( n \) it results:

\[
\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)} > \frac{1}{k} (\ln \ln x_{n+1} - \ln \ln x_1).
\]

Because \( \lim_{m \to \infty} x_m = \infty \) we have \( \lim_{m \to \infty} \ln \ln x_m = \infty \), i.e., the series:

\[
\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}
\]

is divergent. The Proposition 1 is proved.

**Proposition 2.** Series \( \sum_{n=2}^{\infty} \frac{1}{S(n)} \), where \( S \) is the Smarandache function, is divergent.

**Proof.** We use Proposition 1 for \( x_n = n \).

**Remarks.**
1) If \( x_n \) is the \( n \)-th prime number, then the series \( \sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)} \) is divergent.
2) If the sequence \( (x_n)_{n \geq 1} \) forms an arithmetical progression of natural numbers, then the series \( \sum_{n=1}^{\infty} \frac{1}{S(x_n)} \) is divergent.

3) The series \( \sum_{n=1}^{\infty} \frac{1}{S(2n+1)} \), \( \sum_{n=1}^{\infty} \frac{1}{S(4n+1)} \) etc., are all divergent.
In conclusion. Proposition 1 offers us an unitary method to prove that the series having one of the preceding aspects are divergent.

**Proposition 3.** The series:

\[ \sum_{n=2}^{\infty} \frac{1}{S(2) \cdot S(3) \cdots S(n)} \]

where S is the Smarandache function, is convergent to a number \( s \in (71/100, 101/100) \).

**Proof.** From the definition of the Smarandache function it results \( S(n) \leq n! \), \((\forall)n \in \mathbb{N}^* \setminus \{1\}, so \frac{1}{S(n)} \geq \frac{1}{n!}\).

Summing up, beginning with \( n = 2 \) we obtain:

\[ \sum_{n=2}^{\infty} \frac{1}{S(2) \cdot S(3) \cdots S(n)} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2. \]

The product \( S(2) \cdot S(3) \cdots S(n) \) is greater than the product of prime numbers from the set \( \{1, 2, \ldots, n\} \), because \( S(p) = p \), for \( p \) a prime number. Therefore:

\[ \frac{1}{\prod_{i=2}^{n} S(i)} < \frac{1}{\prod_{i=1}^{k} p_i}, \quad (7) \]

where \( p_k \) is the biggest prime number smaller or equal to \( n \).

There are the inequalities:

\[ S = \sum_{n=2}^{\infty} \frac{1}{S(2)S(3) \cdots S(n)} = \frac{1}{S(2)} + \frac{1}{S(2)S(3)} + \frac{1}{S(2)S(3)S(4)} + \cdots + \]

\[ + \frac{1}{S(2)S(3) \cdots S(k)} + \cdots < \frac{1}{2} + \frac{2}{2 \cdot 3} + \frac{2}{2 \cdot 3 \cdot 5} + \frac{4}{2 \cdot 3 \cdot 5 \cdot 7} + \]

\[ + \frac{2}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11} + \cdots + \frac{p_{k+1} - p_k}{p_1p_2 \cdots p_k} + \cdots \quad (8) \]

Using the inequality \( p_1p_2 \cdots p_k > p_{k+1}^3 \), \((\forall)k \geq 5 \) [2], we obtain:
\( S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{1}{p_6} + \frac{1}{p_7} + \ldots + \frac{1}{p_{k+1}} \) (9)

We note \( P = \frac{1}{p_6} + \frac{1}{p_7} + \ldots \) and observe that \( P < \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} + \ldots \).

It results:

\[ P < \frac{\pi^2}{6} - \left( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{12^2} \right) \]

where

\[ \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \quad \text{(EULER)}. \]

Introducing in (9) we obtain:

\[ S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{\pi^2}{6} - 1 - \frac{1}{2^2} - \frac{1}{3^2} - \ldots - \frac{1}{12^2}. \]

Estimating with an approximation of an order not more than \( \frac{1}{10^2} \), we find:

\[ 0.71 \leq \sum_{n=2}^{\pi} \frac{1}{S(2)S(3)\ldots S(n)} < 1.01. \] (10)

The Proposition 3 is proved.

Remark. Giving up at the right increase from the first terms in the inequality (8) we can obtain a better right framing:

\[ \sum_{n=2}^{\pi} \frac{1}{S(2)S(3)\ldots S(n)} < 0.97. \] (11)

Proposition 4. Let \( \alpha \) be a fixed real number, \( \alpha \geq 1 \). Then the series

\[ \sum_{n=2}^{\pi} \frac{n^\alpha}{S(2)S(3)\ldots S(n)} \]

is convergent (fourth constant of Smarandache).

Proof. Be \( (p_k)_{k \geq 1} \) the sequence of prime numbers. We can write:
\[
\begin{align*}
\frac{2^\alpha}{S(2)} &= \frac{2^\alpha}{2} = 2^{\alpha-1} \\
\frac{3^\alpha}{S(2)S(3)} &= \frac{3^\alpha}{p_1p_2} \\
\frac{4^\alpha}{S(2)S(3)S(4)} &= \frac{4^\alpha}{p_1p_2} < \frac{p_3^\alpha}{p_1p_2} \\
\frac{5^\alpha}{S(2)S(3)S(4)S(5)} &= \frac{5^\alpha}{p_1p_2p_3} < \frac{p_4^\alpha}{p_1p_2p_3} \\
\frac{6^\alpha}{S(2)S(3)S(4)S(5)S(6)} &= \frac{6^\alpha}{p_1p_2p_3} < \frac{p_4^\alpha}{p_1p_2p_3}
\end{align*}
\]

\[
\begin{align*}
\frac{n^\alpha}{S(2)S(3)\cdots S(n)} &< \frac{n^\alpha}{p_1p_2\cdots p_k} < \frac{p_{k+1}^\alpha}{p_1p_2\cdots p_k}
\end{align*}
\]

where \(p_i \leq n, i \in \{1, \ldots, k\}, p_{k+1} > n.\)

Therefore
\[
\sum_{i=1}^{\infty} \frac{n^\alpha}{S(2)S(3)\cdots S(n)} < 2^{\alpha-1} + \sum_{k=1}^{\infty} \frac{(p_{k+1} - p_k) \cdot p_{k+1}^\alpha}{p_1p_2\cdots p_k} < 2^{\alpha-1} + \sum_{k=1}^{\infty} \frac{p_{k+1}^\alpha}{p_1p_2\cdots p_k}.
\]

Then it exists \(k_0 \in \mathbb{N}\) such that for any \(k \geq k_0\) we have:

\(p_1p_2\cdots p_k > p_{k+1}^{\alpha+3}\).

Therefore
\[
\sum_{i=1}^{\infty} \frac{n^\alpha}{S(2)S(3)\cdots S(n)} < 2^{\alpha-1} + \sum_{k=1}^{k_0-1} \frac{p_{k+1}^\alpha}{p_1p_2\cdots p_k} + \sum_{k=k_0}^{\infty} \frac{1}{2^{k+1}}.
\]
Because the series $\sum_{k=k_0}^{\infty} \frac{1}{p_{k+1}}$ is convergent it results that the given series is convergent too.

Consequence 1. It exists $n_0 \in \mathbb{N}$ so that for each $n \geq n_0$ we have $S(2)S(3) \ldots S(n) > n^n$.

Proof. Because $\lim_{n \to \infty} \frac{n^n}{S(2)S(3) \ldots S(n)} = 0$, there is $n_0 \in \mathbb{N}$ so that

$\frac{n^n}{S(2)S(3) \ldots S(n)} < 1$ for each $n \geq n_0$.

Consequence 2. It exists $n_0 \in \mathbb{N}$ so that:

$S(2) + S(3) + \ldots + S(n) > (n - 1) \cdot n^{\frac{n}{n-1}}$ for each $n \geq n_0$.

Proof. We apply the inequality of averages to the numbers $S(2), S(3), \ldots, S(n)$:

$S(2) + S(3) + \ldots + S(n) > (n - 1) \cdot \sqrt[n-1]{S(2)S(3) \ldots S(n)} > (n - 1) n^{\frac{n}{n-1}}, \; \forall n \geq n_0$.

REFERENCES


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