The aim of this article is to introduce two functions and to give some simple properties for one of them. The function's properties are studied in connection with the prime numbers. Finally, these functions are applied to obtain some inequalities concerning the Smarandache's function.

1. Introduction

In this section, the main results concerning the Smarandache and Euler's functions are review. Smarandache proposed [1980] a function $S : N^* \rightarrow N$ defined by $S(n) = \min\{k,k! \mid n\}$. This function satisfies the following main equations:

1. $(n,m) = 1 \Rightarrow S(n \cdot m) = \min\{S(n),S(m)\}$

2. $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_r^{k_r} \Rightarrow S(n) = \min\{S(p_1^{k_1}),S(p_2^{k_2}),\ldots,S(p_r^{k_r})\}$

3. $(\forall \, n > 1) \, S(n) \leq n$

and the equality in the inequality (3) is obtained if and only if $n$ is a prime number. The research on the Smarandache's function has been carried out in several directions. One of these direction studies the average function $\bar{S} : N^* \rightarrow N$ defined by $\bar{S}(n) = \frac{\sum_{i=1}^{n} S(i)}{n}$. Tabirca [1997] gave the following two upper bounds for this function $(\forall \, n > 5) \, \bar{S}(n) \leq \frac{3}{8} \cdot n - \frac{1}{4} - \frac{2}{n}$ and $(\forall \, n > 23) \, \bar{S}(n) \leq \frac{21}{72} \cdot n + \frac{1}{12} - \frac{2}{n}$ and conjectured that $(\forall \, n > 1) \, \bar{S}(n) \leq \frac{2}{\ln n}$.

Let $\phi : N^* \rightarrow N$ be the Euler function defined by $\phi(n) = \text{card}\{k = 1,n(k,n) = 1\}$. The main properties of this function are review below:
1. \( (n,m) = 1 \Rightarrow \varphi(n \cdot m) = \varphi(n) \cdot \varphi(m) \)

   (4)

2. \( n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_s^{k_s} \Rightarrow \varphi(n) = n \cdot \prod_{i=1}^{s} \left(1 - \frac{1}{p_i}\right) \)

   (5)

3. \( \varphi\left(\frac{n}{m}\right) = \text{card}\{k = 1, n(k,n) = m\}. \)

   (6)

   It is known that if \( f : \mathbb{N}^* \rightarrow \mathbb{N} \) is a multiplicative function then the function \( g : \mathbb{N}^* \rightarrow \mathbb{N} \)

   defined by \( g(n) = \sum_{d \mid n} f(d) \) is multiplicative as well.

2. The functions \( \psi_1, \psi_2 \)

   In this section two functions are introduced and some properties concerning them are presented.

   Definition 1.

   Let \( \psi_1, \psi_2 \) be the functions defined by the formulas

1. \( \psi_1 : \mathbb{N}^* \rightarrow \mathbb{N}, \psi_1(n) = \sum_{i=1}^{n} \frac{n}{(i,n)} \)

   (7)

2. \( \psi_2 : \mathbb{N}^* \rightarrow \mathbb{N}, \psi_2(n) = \sum_{i=1}^{n} \frac{i}{(i,n)} \)

   (8)

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Table 1. Table of the functions $\psi_1, \psi_2$.

Remarks 1.
1. These functions are correctly defined based on the implication

$$\frac{n}{(i,n)}, \frac{i}{(i,n)} \in \mathbb{N} \Rightarrow \sum_{i=1}^{n} \frac{n}{(i,n)} \sum_{i=1}^{n} \frac{i}{(i,n)} \in \mathbb{N}.$$

2. If $p$ is prime number, then the equations $\psi_1(p) = p^2 - p + 1$ and $\psi_2(p) = \frac{p(p-1)}{2}$ can be easily verified.

3. The values of these functions for the first 30 natural numbers are shown in Table 1. From this table, it is observed that the values of $\psi_1$ are always odd and moreover the equation

$$\psi_2(n) = \frac{\psi_1(n)}{2}$$

seems to be true.

Proposition 1 establishes a connection between $\psi_1$ and $\varphi$.

**Proposition 1**

If $n > 0$ is an integer number, then the equation

$$\psi_1(n) = \sum_{d \mid n} d \cdot \varphi(d)$$  \hspace{1cm} (9)

holds.

**Proof**

Let $A_d = \{\ell = 1, n \mid (i,n) = d\}$ be the set of the elements which satisfy $(i,n) = d$. The following transformations of the function $\psi_1$ hold.
Csing(6) the equation (10) gives $\psi_1(n) = \frac{\sum_{d|n} \phi(d)}{n}$.

Changing the index of the last sum, the equation (9) is found true.

The function $g(n) = n\phi(n)$ is multiplicative resulting in that the function $\psi_1(n) = \sum_{d|n} d \cdot \phi(d)$ is multiplicative. Therefore, it is sufficient to find a formula for $\psi_1(p^k)$, where $p$ is a prime number.

**Proposition 2.**

If $p$ is a prime number and $k \geq 1$ then the equation

$$\psi_1(p^k) = \frac{p^{2^{k-1}} - 1}{p - 1}$$

holds.

**Proof**

The equation (11) is proved based on a direct computation, which is described below.

$$\psi_1(p^k) = \sum_{d|p^k} d \cdot \phi(d) = 1 - \sum_{i=1}^{k} p^i \cdot \phi(p^i) = 1 - \sum_{i=1}^{k} p^i =$$

$$= 1 - \sum_{i=1}^{k} \frac{p^{2^i} - 1}{p^{2^i} - 1} = 1 - \sum_{i=1}^{k} \frac{p^{2^i} - 1}{p - 1} = \frac{p^{2^{k-1}} - 1}{p - 1}$$

Therefore, the equation (11) is true.

**Theorem 1.**

If $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_s^{k_s}$ is the prime numbers decomposition of $n$, then the formula

$$\psi_1\left(\prod_{i=1}^{s} p_i^{k_i}\right) = \prod_{i=1}^{s} \frac{p_i^{2^{k_i-1}} - 1}{p_i - 1}$$

holds.

**Proof**

The proof is directly found based on Proposition 1 and on the multiplicative property of $\psi_1$. 

\[ \text{85} \]
Obviously, if $p$ is a prime number then $\psi_1(p) = \frac{p^2 - 1}{p + 1} = p^2 - p - 1$ holds finding again the equation from Remark 1.2. If $n = p_1 \cdot p_2 \cdots p_s$ is a product of prime numbers then the following equation is true.

$$\psi_1(n) = \psi_1(p_1 \cdot p_2 \cdots p_s) = \prod_{i=1}^{s} (p_i^2 - p_i - 1)$$

(13)

**Proposition 3.**

$$(\forall \ n > 1) \quad \sum_{i=1}^{n} i = \frac{n \cdot \varphi(n)}{2}$$

(14)

**Proof**

This proof is made based on the *Inclusion & Exclusion* principle.

Let $D_p = \{i = 1, 2, \ldots, n \mid p\}$ be the set which contains the multiples of $p$.

This set satisfies

$$D_p = p \cdot \Big\{ 1, 2, \ldots, \frac{n}{p} \Big\} \quad \text{and} \quad \sum_{i=1}^{n} i = p \cdot \sum_{i=1}^{\frac{n}{p}} i = p \cdot \frac{n \cdot \left( \frac{n}{p} + 1 \right)}{2} = \frac{n \cdot \left( \frac{n}{p} - 1 \right)}{2}.$$ 

Let $n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_s^{i_s}$ be the prime number decomposition of $n$.

The following intersection of sets

$$D_{p_1} \cap D_{p_2} \cap \cdots \cap D_{p_s} = \{i = 1, 2, \ldots, n \mid p_1 \cdot p_2 \cdot \cdots \cdot p_s \mid n\}$$

is evaluated as follows

$$D_{p_1} \cap D_{p_2} \cap \cdots \cap D_{p_s} = \{i = 1, 2, \ldots, n \mid p_1 \cdot p_2 \cdot \cdots \cdot p_s \mid n\} = D_{p_1 \cdot p_2 \cdot \cdots \cdot p_s}.$$

Therefore, the equation

$$\sum_{i \in D_{p_1} \cap D_{p_2} \cap \cdots \cap D_{p_s}} i = \sum_{i \in D_{p_1} \cap D_{p_2} \cap \cdots \cap D_{p_s}} i = \frac{n \cdot \left( \frac{n}{p_1 \cdot p_2 \cdots p_s} + 1 \right)}{2}$$

(14)

holds.

The *Inclusion & Exclusion* principle is applied based on

$$D = \{i = 1, 2, \ldots, n \mid (i, n) = 1\} = \{1, 2, \ldots, n\} \setminus \bigcup_{i=1}^{s} D_{p_i}$$

and it gives
Applying (14), the equation (15) becomes
\[
\sum_{i=1}^{n} i = \sum_{i=1}^{n} i - \sum_{m=1}^{n} (-1)^{m-1} \sum_{i \in A_{j_1}, \ldots, j_m} \frac{n}{2} \left( \frac{1}{\mathcal{P}_{j_1} \cdot \mathcal{P}_{j_2} \cdots \mathcal{P}_{j_m}} + 1 \right).
\] (16)

The right side of the equation (16) is simplified by reordering the terms as follows
\[
\sum_{i=1}^{n} i = \sum_{i=1}^{n} \left( 1 - \sum_{m=1}^{n} (-1)^{m-1} \sum_{i \in A_{j_1}, \ldots, j_m} \frac{1}{\mathcal{P}_{j_1} \cdot \mathcal{P}_{j_2} \cdots \mathcal{P}_{j_m}} \right) = \frac{n}{2} \left( 1 - \sum_{m=1}^{n} (-1)^{m-1} \sum_{i \in A_{j_1}, \ldots, j_m} \right).
\]

Therefore, the equation (14) holds. 

Obviously, the equation (14) does not hold for \( n=1 \) because
\[\sum_{i=1}^{1} i = 1 \text{ and } \frac{n \cdot \varphi(n)}{2} = \frac{1}{2}.\]

Based on Proposition 3, the formula of the second function is found.

**Proposition 4.**

The following equation
\[ (\forall n > 1) \ \varphi_2(n) = \frac{\varphi_1(n) + 1}{2} \] (17)
holds.

**Proof**

Let \( I_{n,d} = \{ i = 1, 2, \ldots, n \mid (i,n) = d \} \) be the set of indices which satisfy \( (i,n) = d \). Obviously, the following equation
\[ (\forall d \mid n) \ I_{n,d} = d \cdot I_{\frac{n}{d}}, \] (18)
holds. Based on (18), the sum \( \sum_{i=1}^{n} \frac{i}{(i,n)} \) is transformed as follows
\[ \varphi_2(n) = \sum_{i=1}^{n} \frac{i}{(i,n)} = \sum_{d \mid n} d^{-1} \cdot \sum_{i \in I_{n,d}} i = \sum_{d \mid n} d^{-1} \cdot \sum_{i \in I_{n,d}} i = \sum_{d \mid n} d^{-1} \cdot \sum_{i \in I_{n,d}} i. \] (19)

Proposition 3 is applied for any divisor \( d \mid n \) and the equation (19) becomes
Completing the last sum and changing the index, the equation (20) is transformed as follows

\[
\psi_2(n) = 1 + \frac{1}{2} \sum_{d \mid n} d \cdot \varphi(d) = 1 - \frac{1}{2} \sum_{d \mid n} d \cdot \varphi(d) = \frac{1}{2} - \frac{1}{2} \psi_1(n)
\]

resulting in that (17) is true.

Remarks 2

1. Based on the equation \( \psi_2(n) = \frac{\psi_1(n) + 1}{2} \), it is found that \( \psi_1(n) = 2 \cdot \psi_2(n) - 1 \) is always an odd number and that the equation \( \psi_2(n) = \left\lfloor \frac{\psi_1(n)}{2} \right\rfloor \) holds.

2. If \( n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_s^{k_s} > 1 \) is the prime numbers decomposition of \( n \), then the formula

\[
\psi_2(\prod_{i=1}^{s} p_i^{k_i}) = \frac{1}{2} - \frac{1}{2} \sum_{i=1}^{s} \frac{p_i^{k_i+1}}{p_i+1}
\]

holds.

3. Upper bounds for the Smarandache's function

In this section, an application of the functions \( \psi_1, \psi_2 \) is presented. Based on these function an inequality concerning the Smarandache's function is proposed and some upper bounds for \( \overline{S}(n) = \frac{1}{n} \sum_{i=1}^{n} S(i) \) are deduced.

Let \( p_1 = 2, p_2 = 3, \ldots, p_m, \ldots \) be the set of the prime numbers.

Proposition 5.

\[
(\forall i \geq p_m) \left( \forall j = 1, p_1 \cdot p_2 \cdots p_m \right) S(p_1 \cdot p_2 \cdots p_m \cdot i + j) \leq \frac{p_1 \cdot p_2 \cdots p_m \cdot i + j}{(p_1 \cdot p_2 \cdots p_m, j)}
\]

Proof

Let \( i, j \) be two numbers such that \( i \geq p_m \) and \( j = 1, p_1 \cdot p_2 \cdots p_m \).

Let us suppose that \( (p_1 \cdot p_2 \cdots p_m, j) = p_{i_1} \cdot p_{i_2} \cdots p_{i_v} \) and \( j = p_{j_1} \cdot p_{j_2} \cdots p_{j_u} \cdot f_1 \).
Based on the inequality \( p_1 \cdot p_2 \cdots p_m \cdot i + j_1 \geq i + 1 \geq p_m + 1 \), we find that the product

\[
\left( \frac{p_1 \cdot p_2 \cdots p_m \cdot i + j_1}{p_1 \cdot p_2 \cdots p_m} \right)!
\]

contains the factors \( p_1, p_2, \ldots, p_i \) and

\[
\frac{p_1 \cdot p_2 \cdots p_m \cdot i + j_1}{p_1 \cdot p_2 \cdots p_m}
\]

therefore, the inequality (21) is found true.

Proposition 6.

\[
(\forall i \geq p_m) \sum_{j=1}^{p_1 \cdot p_2 \cdots p_m} S\left(p_1 \cdot p_2 \cdots p_m \cdot i + j \right) \leq i \cdot \psi_1\left(p_1 \cdot p_2 \cdots p_m\right) + \psi_2\left(p_1 \cdot p_2 \cdots p_m\right) \tag{22}
\]

Proof

The equation (21) is applied for this proof as follows:

\[
\sum_{j=1}^{p_1 \cdot p_2 \cdots p_m} S\left(p_1 \cdot p_2 \cdots p_m \cdot i + j \right) \leq \sum_{j=1}^{p_1 \cdot p_2 \cdots p_m} \frac{p_1 \cdot p_2 \cdots p_m \cdot j}{p_1 \cdot p_2 \cdots p_m + j}
\]

\[
= i \cdot \sum_{j=1}^{p_1 \cdot p_2 \cdots p_m} \frac{p_1 \cdot p_2 \cdots p_m}{p_1 \cdot p_2 \cdots p_m + j} + \sum_{j=1}^{p_1 \cdot p_2 \cdots p_m} \frac{j}{p_1 \cdot p_2 \cdots p_m + j}
\]

Applying the definitions of the functions \( \psi_1, \psi_2 \), the inequality (22) is found true.

Theorem 2.

The following inequality

\[
\overline{S}(n) = \frac{1}{n} \sum_{i=1}^{n} S(i) \leq \frac{\psi_1\left(p_1 \cdot p_2 \cdots p_m\right)}{2} \cdot n + \frac{1}{2} \cdot \psi_2\left(p_1 \cdot p_2 \cdots p_m\right) \cdot \frac{1}{n} \cdot n \cdot \frac{1}{C_m} \tag{23}
\]

is true for all \( n > p_1 \cdot p_2 \cdots p_{m-1} \cdot p_m \), where

\[
C_m = \sum_{i=1}^{p_1 \cdot p_2 \cdots p_m} S(i) - \psi_1\left(p_1 \cdot p_2 \cdots p_m\right) \cdot \frac{p_m - 1}{2} - \psi_2\left(p_1 \cdot p_2 \cdots p_m\right) \cdot (p_m - 1) \tag{24}
\]

is a constant which does not depend on \( n \).
Proof

Proposition 6 is used for this proof.

Let $n$ be a number such that $n > p_1 \cdot p_2 \cdots p_{m-1} \cdot p_m^2$. The sum $\sum_{i=1}^{n} S(i)$ is split into two sums as follows

$$\sum_{i=1}^{n} S(i) = \sum_{i=1}^{n} S(i) + \sum_{i=p_1 \cdot p_2 \cdots p_{m-1} \cdot p_m^2}^{n} S(i) \leq \sum_{i=p_1 \cdot p_2 \cdots p_{m-1} \cdot p_m^2}^{n} S(i) = \sum_{i=p_1 \cdot p_2 \cdots p_{m-1} \cdot p_m^2}^{n} S(i) = \sum_{i=p_1 \cdot p_2 \cdots p_{m-1} \cdot p_m^2}^{n} S(i) + \sum_{i=p_1 \cdot p_2 \cdots p_{m-1} \cdot p_m^2}^{n} S(i).$$

For the second sum the inequality (22) is applies resulting in the following inequality

$$\sum_{i=1}^{n} S(i) \leq \sum_{i=1}^{n} S(i) + \sum_{i=p_1 \cdot p_2 \cdots p_{m-1} \cdot p_m^2}^{n} \left[ \psi_1(p_1 \cdot p_2 \cdots p_m) + \psi_2(p_1 \cdot p_2 \cdots p_m) \right].$$

Calculating the last sum, the inequality (25) becomes

$$\sum_{i=1}^{n} S(i) \leq \sum_{i=1}^{n} S(i) + \sum_{i=p_1 \cdot p_2 \cdots p_{m-1} \cdot p_m^2}^{n} \left[ \psi_1(p_1 \cdot p_2 \cdots p_m) \left( \frac{n}{p_1 \cdot p_2 \cdots p_m} - 1 \right) \right] + \sum_{i=p_1 \cdot p_2 \cdots p_{m-1} \cdot p_m^2}^{n} \left[ \psi_2(p_1 \cdot p_2 \cdots p_m) - \psi_3(p_1 \cdot p_2 \cdots p_m) \right].$$

Based on the double inequality $\frac{n}{p_1 \cdot p_2 \cdots p_m} - 1 < \frac{n}{p_1 \cdot p_2 \cdots p_m} \leq \frac{n}{p_1 \cdot p_2 \cdots p_m}$, we find

$$\sum_{i=1}^{n} S(i) \leq \psi_1(p_1 \cdot p_2 \cdots p_m) \left( \frac{n}{p_1 \cdot p_2 \cdots p_m} - 1 \right) + \frac{1}{2} \psi_2(p_1 \cdot p_2 \cdots p_m) + \psi_3(p_1 \cdot p_2 \cdots p_m).$$

Dividing by $n$ and using Proposition 4, the equation (22) is found true.

4. Conclusions

The inequality (22) extends the results presented by Tabirca [1997] and generates several inequalities concerning the function $S$, which are presented in the following:
\( m = 1 \Rightarrow (n > 4) \quad \sum_{i=1}^{n} S(i) \leq 0.375 \cdot n + 0.75 - \frac{5}{n} \)

\( m = 2 \Rightarrow (n > 18) \quad \sum_{i=1}^{n} S(i) \leq 0.29167 \cdot n + 1.76 + \frac{24}{n} \)

\( m = 3 \Rightarrow (n > 150) \quad \sum_{i=1}^{n} S(i) \leq 0.245 \cdot n - 7.35 - \frac{1052}{n} \)

\( m = 4 \Rightarrow (n > 1470) \quad \sum_{i=1}^{n} S(i) \leq 0.215 \cdot n - 45.15 - \frac{176859}{n} \)

The coefficients of \( n \) from the above inequalities are decreasing and the inequalities are stronger and stronger. Therefore, it is natural to investigate other upper bounds such as the bound proposed by Tabirca [1997] \( \sum_{i=1}^{n} S(i) \leq \frac{2 \cdot n}{\ln n} \). Ibstedt based on an LBASIC program [Ibstedt 1997] proved that the inequality \( \sum_{i=1}^{n} S(i) \leq \frac{n}{\ln n} \) holds for natural numbers less than 5000000. A proof for this result has not been found yet.

References