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<td>6</td>
<td>6 7 4 5</td>
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<td>14 15 12 13</td>
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<td>7 4 5 6</td>
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<td>15 12 13 14</td>
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<td>7 4 5 6</td>
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</tr>
</tbody>
</table>
The table on the first cover represents a Smarandache Group under the operation \( /+\) [Talukdar, 19].


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In year 2000 our journal celebrates 10 years of existence.
Abstract:

The set of p different equivalence classes is \( Z_p = \{ [0], [1], [2], \ldots, [k], \ldots, [p-1] \} \). For convenience, we have omitted the brackets and written \( k \) in place of \([k]\). Thus \( Z_p = \{ 0, 1, 2, \ldots, k, \ldots, p-1 \} \).

The elements of \( Z_p \) can be written uniquely as \( m \)-adic numbers. If \( r = (a_{n-1}a_{n-2} \ldots a_0)_m \) and \( s = (b_{n-1}b_{n-2} \ldots b_0)_m \) be any two elements of \( Z_p \), then \( r \Delta s \) is defined as 
\[
\left| |a_{n-1} - b_{n-1}|, |a_{n-2} - b_{n-2}|, \ldots, |a_0 - b_0| \right|_m
\]
then \( (Z_p, \Delta) \) is a groupoid known as SMARANDACHE GROUPOID. If we define a binary relation \( r \equiv s \Leftrightarrow r \Delta C(r) = s \Delta C(s) \), where \( C(r) \) and \( C(s) \) are the complements of \( r \) and \( s \) respectively, then we see that this relation is equivalence relation and partitions \( Z_p \) into some equivalence classes. The equivalence class \( D_{\text{aux}(Z_p)} = \{ r \in Z_p : r \Delta C(r) = \text{Sup}(Z_p) \} \) is defined as \( D \)-form. Properties of SMARANDACHE GROUPOID and \( D \)-form are discussed here.

Key Words: SMARANDACHE GROUPOID, complement element and \( D \)-form.

1. Introduction:

Let \( m \) be a positive integer greater than one. Then every positive integer \( r \) can be written uniquely in the form \( r = a_{n-1}m^{n-1} + a_{n-2}m^{n-2} + \ldots + a_1m + a_0 \) where \( n \geq 0, a_i \) is an integer, \( 0 \leq a_i < m \), \( m \) is called the base of \( r \), which is denoted by \( (a_{n-1}a_{n-2} \ldots a_0)_m \). The absolute difference of two integers \( r = (a_{n-1}a_{n-2} \ldots a_0)_m \) and \( s = (b_{n-1}b_{n-2} \ldots b_0)_m \) denoted by \( r \Delta s \) and defined as
\[
r \Delta s = (|a_{n-1} - b_{n-1}|, |a_{n-2} - b_{n-2}|, \ldots, |a_0 - b_0|)_m
\]
\[
= (c_{n-1}c_{n-2} \ldots c_1c_0)_m, \quad \text{where } c_i = |a_i - b_i| \text{ for } i = 0, 1, 2, \ldots, n-1.
\]
In this operation, \( r \Delta s \) is not necessarily equal to \( |r - s| \) i.e. absolute difference of \( r \) and \( s \).

If the equivalence classes of \( Z_p \) are expressed as \( m \)-adic numbers, then \( Z_p \) with binary operation \( \Delta \) is a groupoid, which contains some non-trivial groups. This groupoid is defined as SMARANDACHE GROUPOID. Some properties of this groupoid are established here.

2. Preliminaries:

We recall the following definitions and properties to introduce SMARANDACHE GROUPOID.
Definition 2.1 (2)

Let \( p \) be a fixed integer greater than one. If \( a \) and \( b \) are integers such that \( a-b \) is divisible by \( p \), then \( a \) is congruent to \( b \) modulo \( p \) and indicate this by writing \( a \equiv b \pmod{p} \). This congruence relation is an equivalence relation on the set of all integers.

The set of \( p \) different equivalence classes is \( \mathbb{Z}_p = \{ 0, 1, 2, 3, \ldots, p-1 \} \).

Proposition 2.2 (1)

If \( a \equiv b \pmod{p} \) and \( c \equiv d \pmod{p} \), then

i) \( a + c = b + d \pmod{p} \)

ii) \( a \times c = b \times d \pmod{p} \)

Proposition 2.3 (2)

Let \( m \) be a positive integer greater than one. Then every integer \( r \) can be written uniquely in the form

\[
r = a_{n-1}m^{n-1} + a_{n-2}m^{n-2} + \ldots + a_1m + a_0
\]

Where \( n \geq 0 \), \( a_i \) is an integer \( 0 \leq a_i < m \). Here \( m \) is called the base of \( r \), which is denoted by \( (a_{n-1}a_{n-2} \ldots a_1a_0)_{m} \).

Proposition 2.4

If \( r = (a_{n-1}a_{n-2} \ldots a_1a_0)_{m} \) and \( s = (b_{n-1}b_{n-2} \ldots b_1b_0)_{m} \), then

i) \( r = s \) if and only if \( a_i = b_i \) for \( i = 0, 1, 2, \ldots, n-1 \).

ii) \( r < s \) if and only if \( (a_{n-1}a_{n-2} \ldots a_1a_0)_{m} < (b_{n-1}b_{n-2} \ldots b_1b_0)_{m} \)

iii) \( r > s \) if and only if \( (a_{n-1}a_{n-2} \ldots a_1a_0)_{m} > (b_{n-1}b_{n-2} \ldots b_1b_0)_{m} \)

3. Smarandache groupoid:

Definition 3.1

Let \( r = (a_{n-1}a_{n-2} \ldots a_1a_0)_{m} \) and \( s = (b_{n-1}b_{n-2} \ldots b_1b_0)_{m} \), then the absolute difference denoted by \( \Delta \) of \( r \) and \( s \) is defined as

\[
\Delta s = (c_{n-1}c_{n-2} \ldots c_1 c_0)_{m'}
\]

where \( c_i = |a_i - b_i| \) for \( i = 0, 1, 2, \ldots, n-1 \).

Here, \( r \Delta s \) is not necessarily equal to \( |r - s| \). For example

\[
5 = (101)_2 \quad \text{and} \quad 6 = (110)_2 \quad \text{and} \quad 5 \Delta 6 = (011)_2 = 3 \quad \text{but} \quad |5 - 6| = 1.
\]

In this paper, we shall consider \( 5 \Delta 6 = 3 \), not \( 5 \Delta 6 = 1 \).

Definition 3.2

Let \( (\mathbb{Z}_p, +p) \) be a commutative group of order \( p = m^e \). If the elements of \( \mathbb{Z}_p \) are...
expressed as \( m \)-adic numbers as shown below:

\[
\begin{align*}
0 &= (00 \ldots 00)_m \\
1 &= (00 \ldots 01)_m \\
2 &= (00 \ldots 02)_m \\
&\vdots \\
m - 1 &= (00 \ldots 0 m-1)_m \\
m &= (00 \ldots 1 0)_m \\
m + 1 &= (00 \ldots 1 1)_m \\
&\vdots \\
m^2 - 1 &= (00 \ldots m-1 m-1)_m \\
m^2 &= (00 \ldots 1 0 0)_m \\
&\vdots \\
m^n - 1 &= (m-1 m-1 \ldots m-1 m-1)_m
\end{align*}
\]

The set \( \mathbb{Z}_p \) is closed under binary operation \( \Delta \). Thus \( (\mathbb{Z}_p, \Delta) \) is a groupoid. The elements \((00 \ldots 00)_m\) and \((m-1 m-1 \ldots m-1 m-1)_m\) are called infimum and supremum of \( \mathbb{Z}_p \).

The set \( \mathcal{H}_i \) of the elements noted below:

\[
\begin{align*}
0 &= (00 \ldots 00)_m \\
1 &= (00 \ldots 01)_m \\
m &= (00 \ldots 1 0)_m \\
m + 1 &= (00 \ldots 1 1)_m \\
&\vdots \\
m^{n-1} - m &= (0 1 \ldots 1 0)_m = \alpha \text{ (say)} \\
m^n - 1 &= (0 1 \ldots 1 1)_m = \beta \text{ (say)} \\
m^2 - m &= (1 1 \ldots 1 0)_m = \gamma \text{ (say)} \\
m^2 - 1 &= (1 1 \ldots 1 1)_m = \delta \text{ (say)}
\end{align*}
\]

is a proper subset of \( \mathbb{Z}_p \).
is a group of order $2^n$ and its group table is as follows:

<table>
<thead>
<tr>
<th>Δ</th>
<th>0</th>
<th>1</th>
<th>m</th>
<th>m+1</th>
<th>...</th>
<th>...</th>
<th>α</th>
<th>β</th>
<th>γ</th>
<th>δ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>m</td>
<td>m+1</td>
<td>...</td>
<td>...</td>
<td>α</td>
<td>β</td>
<td>γ</td>
<td>δ</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>m+1</td>
<td>m</td>
<td>...</td>
<td>...</td>
<td>β</td>
<td>α</td>
<td>δ</td>
<td>γ</td>
</tr>
<tr>
<td>m</td>
<td>m</td>
<td>m+1</td>
<td>0</td>
<td>1</td>
<td>...</td>
<td>...</td>
<td>γ</td>
<td>δ</td>
<td>α</td>
<td>β</td>
</tr>
<tr>
<td>m+1</td>
<td>m+1</td>
<td>m</td>
<td>1</td>
<td>0</td>
<td>...</td>
<td>...</td>
<td>δ</td>
<td>γ</td>
<td>β</td>
<td>α</td>
</tr>
</tbody>
</table>

Similarly, the proper sub-sets

$H_2 = \{0, 2, 2m, 2(m+1)\} \ldots \{2α, 2β, 2γ, 2δ\}$

$H_3 = \{0, 3, 3m, 3(m+1)\} \ldots \{3α, 3β, 3γ, 3δ\}$

$H_{m-1} = \{0, m-1, m(m-1), m^2-1\} \ldots \{m(m-1)α, (m-1)β, (m-1)γ, (m-1)δ\}$

are groups of order $2^n$ under the operation absolute difference. So the groupoid

$(Z_p, Δ)$ contains mainly the groups $(H_1, Δ), (H_2, Δ), (H_3, Δ), \ldots, (H_{m-1}, Δ)$ and this groupoid is defined as SMARANDACHE GROUPOID. Here we use S.Gd. in place of SMARANDACHE GROUPOID.

**Remarks 3.2**

i) Let $(Z_p, +p)$ be a commutative group of order $p$, where $m^{n-1} < p < m^n$,

then $(Z_p, Δ)$ is not groupoid.

For example $(Z_5, +5)$ is a commutative group of order $5$, where $2^2 < p < 2^3$.

Here $Z_5 = \{0, 1, 2, 3, 4\}$ and

$\begin{align*}
0 &= (0 0 0)_2 \\
1 &= (0 0 1)_2 \\
2 &= (0 1 0)_2 \\
3 &= (0 1 1)_2
\end{align*}$
A composition table of $\mathbb{Z}_p$ is given below:

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<tbody>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>0</td>
</tr>
</tbody>
</table>

Table - 2

Hence $\mathbb{Z}_p$ is not closed under the operation $\Delta$, i.e., $(\mathbb{Z}_p, \Delta)$ is not a groupoid.

i) S. Gd. is not necessarily associative.

Let $\mathbf{1} = (00 \ldots 01)_m$
$\mathbf{2} = (00 \ldots 02)_m$ and
$\mathbf{3} = (00 \ldots 03)_m$ be three elements of $(\mathbb{Z}_p, \Delta)$, then

$(\mathbf{1} \Delta \mathbf{2}) \Delta \mathbf{3} = 2$ and

$\mathbf{1} \Delta (\mathbf{2} \Delta \mathbf{3}) = 0$

i.e., $(\mathbf{1} \Delta \mathbf{2}) \Delta \mathbf{3} \neq \mathbf{1} \Delta (\mathbf{2} \Delta \mathbf{3})$.

ii) S. Gd. is commutative.

iii) S. Gd. has identity element $\mathbf{0} = (00 \ldots 0)_m$

iv) Each element of S. Gd. is self inverse i.e., $\forall p \in \mathbb{Z}_p$, $p \Delta p = 0$.

v) Each element of S. Gd. is self inverse i.e., $\forall p \in \mathbb{Z}_p$, $p \Delta p = 0$.

Proposition 3.3

If $(H, \Delta)$ and $(K, \Delta)$ be two groups of order $2^n$ contained in S. Gd. $(\mathbb{Z}_p, \Delta)$, then H is isomorphic to K.

Proof is obvious.

4. Complement element in S. Gd. $(\mathbb{Z}_p, \Delta)$.

Definition 4.1

Let $(\mathbb{Z}_p, \Delta)$ be a S. Gd., then the complement of any element $p \in \mathbb{Z}_p$ is equal to $p \Delta \text{Sup}(\mathbb{Z}_p) = p \Delta m^n - 1$ i.e., $C(p) = m^n - 1 \Delta p$. This function is known as complement function and it satisfies the following properties.

i) $C(0) = m^n - 1$

ii) $C(m^n - 1) = 0$

iii) $C(C(p)) = p$ $\forall$ $p \in \mathbb{Z}_p$

iv) If $p \leq q$ then $C(p) \geq C(q)$
Definition 4.2

An element $p$ of a S. Gd. $Z_p$ is said to be self complement if $p \Delta \sup(Z_p) = p$ i.e. $C(p) = p$.

If $m$ is an odd integer greater than one, then $\frac{m^n - 1}{2}$ is the self complement of $(Z_p, \Delta)$.

If $m$ is an even integer, then there exists no self complement in $(Z_p, \Delta)$.

Remarks 4.3

i) The complement of an element belonging to a S. Gd. is unique.

ii) The S. Gd. is closed under complement operation.

5. A binary relation in S. Gd.

Definition 5.1

Let $(Z_p, \Delta)$ be a S. Gd. An element $p$ of $Z_p$ is said to be related to $q \in Z_p$ iff $p \Delta C(p) = q \Delta C(q)$ and written as $p \equiv q \Leftrightarrow p \Delta C(p) = q \Delta C(q)$.

Proposition 5.2

For the elements $p$ and $q$ of S. Gd. $(Z_p, \Delta)$, $p \equiv q \Leftrightarrow C(p) \equiv C(q)$.

Proof: By definition

$p \equiv q \Leftrightarrow p \Delta C(p) = q \Delta C(q)$

$\Leftrightarrow C(p) \Delta p = C(q) \Delta q$

$\Leftrightarrow C(p) \Delta C(C(p)) = C(q) \Delta C(C(q))$

$\Leftrightarrow C(p) \equiv C(q)$

Proposition 5.3

Let $(Z_p, \Delta)$ be a S. Gd., then a binary relation $p \equiv q \Leftrightarrow p \Delta C(p) = q \Delta C(q)$ for $p, q \in Z_p$, is an equivalence relation.

Proof: Let $(Z_p, \Delta)$ be a S. Gd. and for any two elements $p$ and $q$ of $Z_p$, let us define a binary relation $p \equiv q \Leftrightarrow p \Delta C(p) = q \Delta C(q)$.

This relation is

i) reflexive for if $p$ is an arbitrary element of $Z_p$, we get $p \Delta C(p) = p \Delta C(p)$ for all $p \in Z_p$. Hence $p \equiv p \Leftrightarrow p \Delta C(p) = p \Delta C(p) \quad \forall \quad p \in Z_p$.

ii) Symmetric, for if $p$ and $q$ are any elements of $Z_p$ such that $p \equiv q$, then $p \equiv q \Leftrightarrow p \Delta C(p) = q \Delta C(q)$

$\Leftrightarrow q \Delta C(q) = p \Delta C(p)$

$\Leftrightarrow q \equiv p$
iii) transitive, for \( p, q, r \) are any three elements of \( \mathbb{Z}_p \) such that
\[
p \equiv q \quad \text{and} \quad q \equiv r, \quad \text{then}
\]
\[
p \equiv q \iff p \Delta C(p) = q \Delta C(q) \quad \text{and}

q \equiv r \iff q \Delta C(q) = r \Delta C(r).
\]
Thus \( p \Delta C(p) = r \Delta C(r) \iff p \equiv r \)

Hence \( p \equiv q \quad \text{and} \quad q \equiv r \) implies \( p \equiv r \)

6. D - Form of S. Gd.

Let \( (\mathbb{Z}_p, \Delta) \) be a S. Gd. of order \( m^n \). Then the equivalence relation referred in the proposition 5.3 partitions \( \mathbb{Z}_p \) into mutually disjoint classes.

Definition 6.1

If \( r \) be any element of S. Gd. \( (\mathbb{Z}_p, \Delta) \) such that \( r \Delta C(r) = x \), then the equivalence class generated by \( x \) is denoted by \( Dx \) and defined by
\[
Dx = \{ r \in \mathbb{Z}_p : r \Delta C(r) = x \}
\]
The equivalence class generated by \( \sup(\mathbb{Z}_p) \) and defined by
\[
D_{\sup(\mathbb{Z}_p)} = \{ r \in \mathbb{Z}_p : r \Delta C(r) = \sup(\mathbb{Z}_p) \}
\]
is called the D - form of \( (\mathbb{Z}_p, \Delta) \).

Example 6.2

Let \( (\mathbb{Z}_9, +) \) be a commutative group, then \( \mathbb{Z}_9 = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8 \} \). If the elements of \( \mathbb{Z}_9 \) are written as 3-adic numbers, then
\[
\mathbb{Z}_9 = \{ (00)_3, (01)_3, (02)_3, (10)_3, (11)_3, (12)_3, (20)_3, (21)_3, (22)_3 \}
\]
and \( (\mathbb{Z}_9, \Delta) \) is a S. Gd. of order \( 3^2 = 9 \). Its composition table is as follows:

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Table - 3

11
Here \( 0 \Delta C(0) = 0 \Delta 8 = 8 \)
\[ 1 \Delta C(1) = 1 \Delta 7 = 6 \]
\[ 2 \Delta C(2) = 2 \Delta 6 = 8 \]
\[ 3 \Delta C(3) = 3 \Delta 5 = 2 \]
\[ 4 \Delta C(4) = 4 \Delta 4 = 0 \]
\[ 5 \Delta C(5) = 5 \Delta 3 = 2 \]
\[ 6 \Delta C(6) = 6 \Delta 2 = 8 \]
\[ 7 \Delta C(7) = 7 \Delta 1 = 6 \]
\[ 8 \Delta C(8) = 8 \Delta 0 = 8 \]

Hence \( D_4 = \{ 0, 2, 6, 8 \} = \{(00)_3, (02)_3, (20)_3, (22)_3 \} \)
\( D_6 = \{ 1, 7 \} \)
\( D_5 = \{ 3, 5 \} \)
\( D_0 = \{ 4 \} \)

The self complement element of \((Z_9, \Delta)\) is 4 and D-form of this S. Gd. is \( \{ 0, 2, 6, 8 \} = D_4 \)

Here \( Z_9 = D_0 \cup D_2 \cup D_6 \cup D_4 \).

**Proposition 6.3**

Any two equivalence classes in a S. Gd. \((Z_p, \Delta)\) are either disjoint or identical.

Proof is obvious.

**Proposition 6.4**

Every S. Gd. \((Z_p, \Delta)\) is equal to the union of its equivalence classes.

Proof is obvious.

**Proposition 6.5**

Every D-form of a S. Gd. \((Z_p, \Delta)\) is a commutative group.

Proof: Let \((Z_p, \Delta)\) be a S. Gd. of order \( P = m^n \). The elements of D-form of this groupoid are as follows.

\[
\begin{align*}
0 &= (00 \ldots 00)_m \\
m - 1 &= (00 \ldots 0 m-1)_m \\
m^2 - m &= (00 \ldots m-1 0)_m \\
m^2 - 1 &= (00 \ldots m-1 m-1)_m \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \\
m^{m-1} - m &= (0 m-1 \ldots m-1 0)_m \\
m^{m-1} - 1 &= (0 m-1 \ldots m-1 m-1)_m \\
m^m - m &= (m-1 m-1 \ldots m-1 0)_m \\
m^m - 1 &= (m-1 m-1 \ldots m-1 m-1)_m
\end{align*}
\]
Here \( (\text{\textit{D}}_{m^n-1}, \Delta) \) is a commutative group and its table is given below:

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\Delta & 0 & m-1 & 0 & 0 & m-2 & m-1 & 0 \\
\hline
0 & 0 & m-1 & 0 & 0 & m-2 & m-1 & 0 \\
\hline
m-1 & m-1 & 0 & m-2 & m-1 & 0 & m-2 & m-1 \\
\hline
m^n-1 & m^n-2 & m^n-3 & m^n-2 & m^n-3 & m^n-2 & m^n-3 & m^n-2 \\
\hline
\end{array}
\]

Table - 4

Remarks 6.6

Let \((\text{\textit{Z}}_p, \Delta)\) be a S. Gd. of order \(m^p\).

The equivalence relation \(p \equiv q \iff p \Delta C(p) = q \Delta C(q)\) partitions \(\text{\textit{Z}}_p\) into some equivalence classes.

i) If \(m\) is odd integer, then the number of elements belonging to the equivalence classes are not equal. In the example 6.2, the number of elements belonging to the equivalence classes \(D_0, D_2, D_6, D_8\) are not equal due to \(m = 3\).

ii) If \(m\) is even integer, then the number of elements belonging to the equivalence classes are equal.

For example, \(\text{\textit{Z}}_{16} = \{0, 1, 2, \ldots, 15\}\) be a commutative group. If the elements of \(\text{\textit{Z}}_{16}\) are expressed as 4- adic numbers, then \((\text{\textit{Z}}_{16}, \Delta)\) is a S. Gd. The composition table of \((\text{\textit{Z}}_{16}, \Delta)\) is given below:
Here $0 \Delta C(0) = 15 = 15 \Delta C(15)$
1 $\Delta C(1) = 13 = 14 \Delta C(14)$
2 $\Delta C(2) = 13 = 13 \Delta C(13)$
3 $\Delta C(3) = 15 = 12 \Delta C(12)$
4 $\Delta C(4) = 7 = 11 \Delta C(11)$
5 $\Delta C(5) = 5 = 10 \Delta C(10)$
6 $\Delta C(6) = 5 = 9 \Delta C(9)$
7 $\Delta C(7) = 7 = 8 \Delta C(8)$

Hence $D_{15} = \{ 0, 3, 12, 15 \}$, $D_{13} = \{ 1, 2, 13, 14 \}$
$D_{5} = \{ 4, 8, 7, 11 \}$, $D_{5} = \{ 5, 6, 9, 10 \}$

The number of elements of the equivalence classes are equal due to $m = 4$, which is even integer.

**Acknowledgement:**

I wish to express my gratitude to Prof. Sashi Sarma, Nalbari and Sjt. Panchanan Sarma, Bidyapur, Nalbari for their encouragement in preparing this paper.
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The notions of the Smarandache group and the Smarandache Boolean ring are introduced here with the help of group action and ring action i.e. module respectively. The centre of the Smarandache groupoid is determined. These are very important for the study of Algebraic structures.

1. The centre of the Smarandache groupoid:

**Definition 1.1**

An element $a$ of the Smarandache groupoid $(Z_p, \Delta)$ is said to be conjugate to $b$ if there exists $r$ in $Z_p$ such that $a = r \Delta b \Delta r$.

**Definition 1.2**

An element $a$ of the Smarandache groupoid $(Z_p, \Delta)$ is called a self conjugate element of $Z_p$ if $a = r \Delta a \Delta r$ for all $r \in Z_p$.

**Definition 1.3**

The set $Z_p^*$ of all self conjugate elements of $(Z_p, \Delta)$ is called the centre of $Z_p$ i.e.

$Z_p^* = \{ a \in Z_p : a = r \Delta a \Delta r \forall r \in Z_p \}$.

**Definition 1.4**

Let $(Z_9, +_9)$ be a commutative group, then $Z_9 = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8 \}$. If the elements of $Z_9$ are written as 3-adic numbers, then $Z_9 = \{ (00)_3, (01)_3, (02)_3, (10)_3, (11)_3, (12)_3, (20)_3, (21)_3, (22)_3 \}$ and $(Z_9, \Delta)$ is a Smarandache groupoid of order 9. Conjugacy relations among the elements of $Z_9$ are determined as follows:
Here \( \mathbb{Z}_9^* \) = \{ 0, 1, 3, 4 \}, the set of all self conjugate elements of \( \mathbb{Z}_9 \) is called the centre of \( (\mathbb{Z}_9^*, \Delta) \). Again \( (\mathbb{Z}_9^*, \Delta) \) is an abelian group.

D-form of the Smarandache groupoid \( (\mathbb{Z}_9^*, \Delta) \) is given by \( D_\Delta = \{ 0, 2, 6, 8 \} \). Again \( (D_\Delta, \Delta) \) is an abelian group. The group table (2) and group table (3) are given below.

\[
\begin{array}{cccc}
0 & 1 & 3 & 4 \\
0 & 0 & 1 & 3 \\
1 & 1 & 0 & 4 \\
3 & 3 & 4 & 0 \\
4 & 4 & 3 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 2 & 6 & 8 \\
0 & 0 & 2 & 6 & 8 \\
2 & 2 & 0 & 8 & 6 \\
6 & 6 & 8 & 0 & 2 \\
8 & 8 & 6 & 2 & 0 \\
\end{array}
\]

Table - 2

Table - 3

From table (2) and table (3), it is clear that \( (\mathbb{Z}_9^*, \Delta) \) and \( (D_\Delta, \Delta) \) are isomorphic to each other.

**Definition 1.3**

The groups \( (\mathbb{Z}_p^*, \Delta) \) and \( (D_{sup(\mathbb{Z}_p)}, \Delta) \) of the Smarandache groupoid \( (\mathbb{Z}_p, \Delta) \) are isomorphic to each other.

Proof is obvious.
2. The Smarandache group:

To introduce the Smarandache group, we have to explain group action on a set. Let $A$ be a group and $B$ a set. An action of $A$ on $B$ is a map, $B \times A \rightarrow B$ written $(b, p) \rightarrow b \Delta p$ such that

i) for every $p, q \in A$ and $b \in B$, we have

$$((b, p), q) = ((b \Delta p) \Delta q) = (b \Delta p) \Delta q = b \Delta (p \Delta q)$$

and

ii) for every $b \in B$, we have $(b, 0) = b \Delta 0 = b$

where $0$ denotes the identity element of the group $A$.

If a group $A$ has an action on $B$, we say that $B$ is a $A$-set or $A$-space. Here in this paper we shall use $B(A)$ in place of "$B$ is a $A$ - space".

Note:

If $B$ is a proper subgroup of $A$, then we get a map $A \times B \rightarrow A$ defined by

$$(a, b) \rightarrow a \Delta b \in A.$$ This is a group action of $B$ on $A$. Then we say that $A$ is a $B$ - set or $B$ - space i.e. $A(B)$ is a $B$ - space. In this paper, by proper subgroup, we mean a group contained in $A$, different from the trivial groups.

Definition 2.1

The smarandache group is defined to be a group $A$ such that $A(B)$ is a $B$ - space, where $B$ is a proper subgroup of $A$.

Examples 2.2

i) The $D$ - form of $(Z_p, \Delta)$ defined by

$$D_{sup(Z_p)} = \{ r \in Z_p : r \Delta C(r) = Sup(Z_p) \} = A$$

is a Smarandache group. If $B$ is a proper subgroup of $A$, then the action of $B$ on $A$ is the map, $A \times B \rightarrow A$ defined by $(a, q) = a \Delta q$ for all $a \in A$ and $q \in B$. This action is a $B$ - action i.e. $A(B)$ is a $B$ - space.

ii) The centre of $(Z_p, \Delta)$ defined by

$$Z_p^* = \{ a \in Z_p : a = r \Delta a \Delta r \quad \forall r \in Z_p \} = A$$

is a Smarandache group. If $B$ be a proper subgroup of $A$, then the action of $B$ on $A$ is the map, $A \times B \rightarrow A$ defined by $(a, p) = a \Delta p$ for all $a \in A$ and for all $p \in B$. This action is a $B$ - space i.e. $A(B)$ is a $B$ - space.

iii) The Addition modulo $m$ of two integers $r = (a_{n-1} a_{n-2} \ldots \ldots a_1 a_0)_m$ and $s = (b_{n-1} b_{n-2} \ldots \ldots b_1 b_0)_m$ denoted by $+/m \backslash \text{and defined as}$

$$r +/m \backslash s = (a_{n-1} a_{n-2} \ldots \ldots a_1 a_0)_m +/m \backslash (b_{n-1} b_{n-2} \ldots \ldots b_1 b_0)_m$$

$$= (a_{n-1} +/m b_{n-1} a_{n-2} +/m b_{n-2} \ldots \ldots a_1 +/m b_1 a_0 +/m b_0)_m$$

$$= (c_{n-1} c_{n-2} \ldots \ldots c_1 c_0)_m,$$ where $c_i = a_i +/m b_i$ for $i = 0, 1, 2, \ldots n-1$. 

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The group $(\mathbb{Z}_p, \cdot_m)$ is a Smarandache group. The group $(\mathbb{Z}_p, \cdot_{m+p})$ contains a proper subgroup $.p\cdot_m$ $(H = \{ 0, 1, 2, 3, \ldots, p-1 \}, \cdot_m)$

Then the action of $H$ on $\mathbb{Z}_p$ is the map $\mathbb{Z}_p \times H \to \mathbb{Z}_p$ defined by $(a, r) = \frac{a}{m} r$ for all $a \in \mathbb{Z}_p$ and for all $r \in H$. This action is a $H$ - space i.e. $\mathbb{Z}_p(H)$ is a $H$ - space.

iv) The set $\mathbb{Z}_{16} = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 \}$ can be written as $\mathbb{Z}_{16} = \{(00)_4, (01)_4, (02)_4, (03)_4, (10)_4, (11)_4, (12)_4, (13)_4, (20)_4, (21)_4, (22)_4, (23)_4, (30)_4, (31)_4, (32)_4, (33)_4 \}$

is a smarandache group under the operation $\cdot_4$ and its group table is as follows:

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Table - 4

From above group table, it is clear that $(H = \{ 0, 1, 2, 3, 8, 9, 10, 11 \}, \cdot_4)$ is a subgroup of $(\mathbb{Z}_{16}, \cdot_4)$. Then the action of $H$ on $\mathbb{Z}_{16}$ is the map $\mathbb{Z}_{16} \times H \to \mathbb{Z}_{16}$ defined by $(a, r) = \frac{a}{4} r$ for all $a \in \mathbb{Z}_{16}$ and for all $r \in H$. This action is a $H$ - space i.e. $\mathbb{Z}_{16}(H)$ is a $H$ - space.

Here $(K = \{ 0, 1, 2, 3 \}, \cdot_4)$ be a subgroup of $(H, \cdot_4)$. Then the action of $K$ on $H$ is the map, $H \times K \to H$ defined by $(b, p) = b \cdot_4 p$ for all $b \in H$ and for all $p \in K$. This action is a $K$ - space i.e. $H(K)$ is a $K$ - space. Hence $H$ is a Smarandache group contained in the Smarandache group $\mathbb{Z}_{16}$. So it is called the Smarandache sub-group.
3. The Smarandache sub-group:

Definition 3.1

The Smarandache sub-group is defined to be a Smarandache group $B$ which is a proper subset of Smarandache group $A$ (with respect to the same induced operation).

4. Smarandache quotient group:

Let $(H, \Delta)$ be Smarandache subgroup of the Smarandache group $(B, \Delta)$, then the quotient group $B/H$ is defined as Smarandache quotient group such that $V(K)$ is a $K$-space, where $K$ is a proper subgroup of $V$ i.e. the group action of $K$ on $V$ is a map $V \times K \rightarrow V$, defined by

\[(H \Delta a), H \Delta p) = (H \Delta a) \Delta (H \Delta p) \text{ for all } H \Delta a \in V \text{ and } H \Delta p \in K\]

5. Smarandache Boolean ring:

Definition 5.1

The intersection of two integers $r = (a_{n-1}a_{n-2} \ldots \ldots a_1a_0)_m$ and $s = (b_{n-1}b_{n-2} \ldots \ldots b_1b_0)_m$ denoted by $r \cap s$ and defined as

\[r \cap s = (a_{n-1} \cap b_{n-1}a_{n-2} \cap b_{n-2} \ldots \ldots a_1 \cap b_1a_0 \cap b_0)_m\]

where $c_i = a_i \cap b_i = \min (a_i, b_i)$ for $i = 0, 1, 2, \ldots \ldots , n-1$

If the equivalence classes of are expressed as $m$-adic numbers, then with binary operation $\cap$ is a groupoid, which contains some non-trivial groups. This groupoid is Smarandache groupoid. Here $(\mathbb{Z}p^+, \Delta, \cap)$ and $(\mathbb{Z}_{\text{sup}(\mathbb{Z}p)}, \Delta, \cap)$ are Boolean rings.

Definition 5.2

The Smarandache Boolean ring is defined to be a Boolean ring $A$ such that the Abelian group $(A, \Delta)$ has both left and right $B$-module, where $B$ is a non-trivial sub-ring of $A$.

From above, we mean an Abelian group $(A, \Delta)$ together with a map, $B \times A \rightarrow A$, written $(b, p) = b \cap p \in A$ such that for every $b, c \in B$ and $p, q \in A$, we have

\[\text{i) } b \cap (p \Delta q) = (b \cap p) \Delta (b \cap q)\]
\[\text{ii) } (b \Delta c) \cap p = (b \cap p) \Delta (c \cap p)\]
\[\text{iii) } (b \cap c) \cap p = b \cap (c \cap p)\]

Again from the map, $A \times B \rightarrow A$, written $(p, b) = p \cap b \in A$ such that for every $p, q \in A$ and $b, c \in B$, we get

\[\text{i) } (p \Delta q) \cap b = (p \cap b) \Delta (q \cap b)\]
\[\text{ii) } p \cap (b \Delta c) = (p \cap b) \Delta (p \cap c)\]
\[\text{iii) } p \cap (b \cap c) = (p \cap b) \cap c\]
Definition 5.3

The Smarandache Boolean sub-ring is defined to be a Smarandache Boolean ring $B$ which is a proper subset of a Smarandache Boolean ring $A$. (with respect to the same induced operation).

Definition 5.4

The Smarandache Boolean ideal is defined to be an ideal $B$ of Smarandache Boolean ring $A$ such that the Abelian group $(C, \Delta)$ has both left and right $B$-module, where $C$ is a proper subset of $B$. From above we mean an Abelian group $(C, \Delta)$ together with a map, $C \times B \rightarrow C$ written $(c, p) = C \cap P \in C$ such that this mapping satisfies all the postulates of both left and right $B$-module.

Examples 5.5

Let $(Z_{256}, +_{256})$ be an Abelian group, then $Z_{256} = \{ 0, 1, 2, \ldots, 255 \}$. If the elements $Z_{256}$ of are written as 4-adic numbers, then

$Z_{256} = \{ (0000), (0001), (0002), (0003), (0010), \ldots, (3333) \}$ and $(Z_{256}, \Delta)$ is a Smarandache groupoid of order 256. The centre of $(Z_{256}, \Delta)$ is

$Z'_{256} = \{ 0, 1, 4, 5, 16, 17, 20, 21, 64, 65, 68, 69, 80, 81, 84, 85 \}$. Here $(Z'_{256}, \Delta, \cap)$ is a Smarandache Boolean ring and its composition tables are given below:

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Table - 5

21
Some Smarandache Boolean sub-rings of \((\mathbb{Z}_{256}, \Delta, \cap)\) are given below:

\begin{align*}
i) & \quad H_1 = \{ 0, 1, 4, 5, 16, 17, 20, 21 \} \\
& \quad H_2 = \{ 0, 1, 4, 5, 64, 65, 68, 69 \} \\
& \quad H_3 = \{ 0, 1, 4, 5, 80, 81, 84, 85 \} \\
& \quad H_4 = \{ 0, 5, 16, 21, 64, 69, 80, 85 \} \\
& \quad H_5 = \{ 0, 1, 16, 17, 64, 65, 80, 81 \} \\
& \quad H_6 = \{ 0, 1, 4, 5 \} \\
& \quad H_7 = \{ 0, 1, 16, 17 \} \\
& \quad H_8 = \{ 0, 1, 64, 65 \} \\
& \quad H_9 = \{ 0, 1, 80, 81 \} \text{ etc.}
\end{align*}

Here Smarandache Boolean subrings \(H_1, H_2, H_3, H_4, H_5, H_9\) are ideals of \((\mathbb{Z}_{256}, \Delta)\).
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FROM BOLYAI'S GEOMETRY TO SMARANDACHE ANTI-GEOMETRY

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ABSTRACT

It is considered the notion of absolute Geometry in its evolution, from the first Non-euclidean Geometry of Lobacewski, Bolyai and Gauss till that of Smarandache Anti-Geometry.

Key words: Euclidean Geometry, Non-euclidean Geometry, Hilbert's axioms and incidence structures deduced from them, Smarandache Geometries, Hjelmslev-Barbilian structures

Any theory or deductive system has two distinguishable parts:
1. the specific part
2. the logical part.

When we formalize one, or both of parts of a theory, we obtain next classification for axiomatical theories:
1. nonformalized,
2. semiformalized, or
3. formalized axiomatical theory.

When J. Bolyai in [3] in 1831, and N. Lobacewski in [12], in 1826 began the studies about non-euclidean Geometries the formalization in mathematics was not yet introduced. Their contributions should be considered as more important as at that moment the formalized system of axioms of Geometry of D. Hilbert was not given.

The way we can establish the metamathematical analyse of a theory are two:
1. sintactical (that is directly)
2. semantical, by the interpretations and models.

J. Bolyai and N. Lobacevski worked only sintactically and not semantically in the study of the metamathematical analyse of their theory.

On the first way, non-contradiction of their Geometry given in [3] and [12] could not be proved, because in a such a way they could not convince that the set of correct
affirmations of their theory were exhausted and that has excluded the possibility to meet a proposition \( p \) correct constructed such that \( p \) and \( \neg p \) to be verified in their Geometry. Later were given semantically the proofs that their non-euclidean Geometry is consistent, and so is non-contradictory, by the models of Bertrami in 1868, after that of Cayley-Klein in 1871 and of H. Poincaré in 1882.

In Bolyai-Lobachevski’s absolute Geometry from a point \( A \) to a line \( a\), \( A \) is not incident with \( a\), there exists a parallel. In this absolute Geometry we have only two possibilities:

1. the Euclidean Geometry
2. the Hyperbolic Geometry.

The elliptical Geometry with none parallel through a point to a line, are excluded from this absolute Geometry. Also this absolute Geometry contained only continuous Geometry.

In 1903 in [6], D. Hilbert proved that the hyperbolic plane geometry can be introduced without to use the tridimensional space, and that is possible to renounce to the axioms of continuity. This is an important moment for research in Geometry because from that moment the notion of absolute Geometry changes its meaning and begins to be different considered to different moments and to different authors. The absolute plane of Bolyai becomes a particular case of the absolute plane in recently researches of Geometry.

From 1889, when D. Hilbert in [5] gave a formalized system of axioms for absolute Geometry, appeared more directions of investigation in Geometry. The incidence structures are largely used and so are introduced a great variety of affine and projective planes and affine and projective spaces.

The great importance of geometrical transformations for geometrical problems was put by F. Klein in "The programme from Erlangen" in 1872, when he began to consider the Geometry as the study of invariant properties to a group of transformations. From that moment the system of axioms of many Geometries are based on the notions of theory of groups. This group is given as an abstract group, and geometrical structure is a consequence of structure of group. This fact was possible, after that it was proved that the geometry can be transposed in the group of its automorphisms generated by axial symmetries. A such a system of axioms is more simple than a classical one, it is easily adopted to the special qualities of non-euclidean Geometries. Compared with a such a system of axioms, the system of axioms of D. Hilbert is more complicated.

As it is the calculus in a field for Analytical Geometry a method of work, as the calculus in the group generated by the axial symmetries becomes a method for proofs in Geometry, after J. Hjelmslev in [7] introduced it. In [16] Thompsen proved that this can become an efficient method of demonstration for the theorems of Euclidean Geometry. This is an attrative method because the hypothesis and conclusions of a theorem can be written simply as relations of group.

The first system of axioms of absolute plane geometry formulated in theory of groups was given by A. Schmidt in [13], and after that F. Bachman in [1]. From that date this method is largely used in Geometry as in [9], [10], [11], [17] and many others works.

In 1954 after E. Sperner gives a group proof of theorem of Desargues for a large classes of Geometries, in absolute geometry are included new-types of geometry, as geometry with centre, with perpendicular nuclei [10], and many others.
The classical Geometries are extended, because are not made hypothesis of order, of continuity or mobility [1], [9], [11], [17].

In 1967 H. Wolf in [21], includes near Euclidean, elliptical, and hyperbolical Geometry, also Minkowski's Geometry.

The geometries constructed over a field of characteristic 2 are included later, by a more general system of axioms of R. Lingenberg in [11].

Another generalization of incidence structures was that in which it were considered geometrical structures to which the line incident with two different points is not unique. A such a theory is consistent and as a model for it we have the Geometry over a ring. Such structures were introduced by J. Hjelmslev in [8] and D. Barbilian in [2]. A new direction of study in Geometry begins from this moment, in which we have also some results.

The researches of absolute Geometry have a natural continuity, the notion of absolute Geometry is a notion in evolution in the modern literature of speciality. This help us to understand better the life, the transformations in the life, and finally this could bring us more wisdom and increasing degree of understanding of human condition, and such to answer to the deep desire of their creatores: that the mathematics to become also a force of life.

Such Florentin Smarandache even in 1969 said that it is natural to consider a new Geometry denying not only one axiom from the axioms of D.H. Hilbert from [5] but more or even all of them, what he did in 1985 in [14] and in 1997 in [15].

So he introduced so called "Smarandache Anti-Geometry". It seems strange but it is natural. We should remember that when J. Bolyai the genial discoverer of first non-Euclidean Geometry was deeply implied in his great work even the great Gauss said that the people are not prepared to receive a new Geometry, a such a new theory. And that was the truth. J. Bolyai suffered very much at that time seeing that he can not be understood, but he was convinced that not only in Mathematics, but in the whole history of thinking his conception represents a crucial point. Besides the value of his discoveries in Mathematics, J. Bolyai must be discovered and then, inevitable loved, as a great thinker preoccupied of the problems of harmonious integration in the life. As we showed in [18], [19] J. Bolyai always was thirsty of harmony and with a stoical wisdom he supported his ideas until the end of his life, a life full of misunderstanding. In spite of all what he met as nonunderstanding he continued to believe in what he created and he felt them to be true.

As J. Bolyai, N. Lobacevski was not understood during his life and his work was not recognized at that time. Their contributions today have to be appreciate even more as at their time the formalized theories has not been introduced.

Beyond the mathematical contribution their works represent an opening meditation of human condition which have not been enough exploited. Feeling the potential of this opening in the understanding of the human complexity we suggested it as a direction of research and to try to imply, we all scientists, to get an amelioration of the human condition as in [18], [19], [20] we did. This research can be done by the utilisation of mathematical ideas and theories to the construction of a model of self-knowledge. Have we ever put the question which are the axioms which stay at the base of the existence? As any theory the human existence should have some axioms, propositions, theorems,
conjectures, false affirmations etc. The consequences of false affirmations in our behaviour can be clearly observed: the pollution of the mind, of the nature, ecological perturbations etc.

We all can realize that the elimination or diminuation of false affirmations about the existence and man, could bring harmony and peace. Taking in consideration the profoundness and credibility of scientists we can hope more and more from us paying attention to this noble work. The incredible technical progress and discoveries of the science have a correspondent in the science of selfknowledge.

The Anti-Geometry introduced by Florentin Smarandache in [14], [15] would correspond to the understanding of the degradation of human condition. Even this "Anti-Geometry" could be a model for this kind of "inner Geometry", in the sense that the degree of degradation represents the different levels of negation of our inner possibilities, of our natural qualities.

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Smarandache's new geometries
a provocation for an ammelioration
of human condition

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ABSTRACT
Are remarked the new Geometries of Smarandache and it is given a relationship and an application of Smarandache Paradoxist Geometry to the ammelioration of human condition by a better understanding of ourselves and of others.

Key words: Non-euclidean Geometry, Bolyai/Lobacewski/Gauss and Riemann Geometry, Smarandache Paradoxist Geometry

In [2], [3], Florentin Smarandache introduced a new type of Geometry. In this Smarandacheian space it is proposed to be considered the theory deduced from the Absolute Geometry of Bolyai and Lobacewski in which the axiom of parallel it is accepted for some pairs of points and lines and it is denied for others. This new Geometry generalizes and unites in the same time: Euclid, Bolyai/Lobacewski/Gauss and Riemann Geometries.

If the first Non-euclidean Geometry introduced by Lobacewski, Bolyai and Gauss surprised the world, such that Gauss said that the people were not prepared to receive a new theory, now we know and accept many kinds of new Geometries. Even in 1969 Florentin Smarandache had put the problem to study a new Geometry in which the parallel from a point to a line to be unique only for some pairs of points and lines and for others: none or more, even infinitely many parallels could be drawn through some points to a line.

Are nowadays people surprise for such new ideas and new Geometries? Certainly not! After then the formalized theories were introduced in Mathematics, a lot of new Geometries could be accepted and semantically to be proved to be non-contradictory by the models created for them as in [1].

In [4] we introduced a new notion for understanding the great diversity of human condition, that of "inner Geometry". Conformly with this notion we differ so much after
the degree of manifestation of our inner possibilities, and from here, after our own blockade of them. To be able to understand and to improve our interhuman relationships these new types of Geometries could help in at least two directions. For a hand, we are in different type of "inner Geometry" from a moment to another moment, and for the other hand: from a person to other one, this "inner Geometry" could be different. In this acceptation we can treat each other with more wisdom, we can find an explanation of so exposed human condition, to be more conscious about the greatness of self knowledge and to imply more in the ammelioration of the existence as a theory in which we want to be with more conciliation. Smarandache's Geometries could be considered in this way, as an important reflection about human condition and his Paradoxist Geometry to find a new model in the theory of existence.

References


The aim of this article is to establish the complexity order of the Erdos function average. This will be studied based on some recent results about the Smarandache function.

1. INTRODUCTION

The main results used in this paper are reviewed in the following. These deal with the main properties of the Smarandache and Erdos functions.

The Smarandache function [Smarandache, 1980] is $S : \mathbb{N}^* \rightarrow \mathbb{N}$ defined by

$$S(n) = \min\{k \in \mathbb{N} | k! = M_n\} \ (\forall n \in \mathbb{N}^*). \ (1)$$

The function $P : \mathbb{N}^* \rightarrow \mathbb{N}$ defined by

$$P(n) = \min\{p \in \mathbb{N} | n = M_p \land p \text{ is prim}\} \ (\forall n \in \mathbb{N}^* \setminus \{1\}), P(1) = 0 \ (2)$$

is named classically the Erdos function. Both functions satisfy the same main properties:

$$(\forall a, b \in \mathbb{N}^*) (a, b) = 1 \Rightarrow S(a \cdot b) = \max\{S(a), S(b)\}, P(a \cdot b) = \max\{P(a), P(b)\}. \ (3)$$

$$(\forall a \in \mathbb{N}^*) P(a) \leq S(a) \leq a \text{ and the equalities occur if } a \text{ is prim}. \ (4)$$

Erdos [1991] found that these two functions have the same values for all most of the natural numbers

$$\lim_{n \to \infty} \left| \left\{ r = 1, n \mid P(i) < S(i) \right\} \right| = 0. \text{ This important result was extended by Ford [1999] to }$$

$$\left| \left\{ r = 1, n \mid P(i) < S(i) \right\} \right| = n \cdot e^{-(\sqrt{3} - \sqrt{2}) \sqrt{\ln n \cdot \ln \ln n}}, \text{ where } \lim_{n \to \infty} a_n = 0. \ (5)$$

Obviously, both functions are neither increasing nor decreasing functions. In this situation, many researchers have tried to study properties concerning their average. Many results that have been published so far deal with complexity orders of the average.

Let us denote $E(f(n)) = \frac{1}{n} \sum_{i=1}^{n} f(n)$ the average of function $f : \mathbb{N}^* \rightarrow \mathbb{R}$. The average $E(S(n))$ was intensively studied by Tabirca [1997, 1998] and Luca [1999]. Tabirca [1998] proved that

$$(\forall n > c_p) E(S(n)) \leq a_p \cdot n + b_p, \text{ where } \lim_{p \to \infty} a_p = \lim_{p \to \infty} b_p = 0. \text{ This means that the order } O(n)$$

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is not properly chosen for $E(S(n))$. Tabirca [1998] conjectured that the average $E(S(n))$ satisfies the equation $E(S(n)) \leq \frac{n}{\ln n}$. Finally and the most important, Luca [1999] proposed the equation

$$\frac{1}{2} \left[ \pi(n) - \pi(\sqrt{n}) \right] < E(S(n)) < \pi(n) + \frac{5}{2} \cdot \ln \ln n + \frac{1}{n} + \frac{31}{5}$$

(7)

where $\pi(x)$ denotes the number of prime numbers less than or equal to $x$. Thus, the complexity order for the average $E(S(n))$ is indeed $O\left(\frac{n}{\log n}\right)$.

2. THE COMPLEXITY ORDER FOR THE ERDOS FUNCTION

Some of the above results are used to find the complexity order of $E(P(n))$. Based on the well-known formula $\lim_{n \to \infty} \frac{\pi(n)}{n} = 1$, Equation (7) gives

$$\frac{1}{2} \leq \lim_{n \to \infty} \inf \frac{E(S(n))}{n} \leq \lim_{n \to \infty} \sup \frac{E(S(n))}{n} \leq 1.$$  

(8)

Theorem 1.

$$\lim_{n \to \infty} \inf \frac{E(S(n))}{n} = \lim_{n \to \infty} \inf \frac{E(P(n))}{n}, \quad \lim_{n \to \infty} \sup \frac{E(S(n))}{n} = \lim_{n \to \infty} \sup \frac{E(P(n))}{n}$$

(9)

Proof Let us denote $A = \{ i = 1, \ldots, n \mid S(i) > P(i) \}$ the set of the numbers that do not satisfy the equation $S(i) = P(i)$. The cardinal of this set is $\mid A \mid = n \cdot \pi(e^{(\sqrt{2} + \epsilon) \sqrt{\ln \ln n}})$, where $\lim_{n \to \infty} a_n = 0$.

The proof is started from the following equation

$$\left| E(S(n)) - E(P(n)) \right| = \frac{1}{n} \left| \sum_{i=1}^{n} S(i) - \sum_{i=1}^{n} P(i) \right| = \frac{1}{n} \left[ \sum_{i \in A} (S(i) - P(i)) \right] \leq \sum_{i \in A} \frac{S(i)}{n}.$$  

(10)

Because we have $\left( \forall i = 1, \ldots, n \right) S(i) \leq n$, Equation (10) gives

$$\left| E(S(n)) - E(P(n)) \right| \leq \left| A \right| = n \cdot \pi(e^{(\sqrt{2} + \epsilon) \sqrt{\ln \ln n}}) \quad \text{and}$$

$$\left| \frac{E(S(n))}{n} - \frac{E(P(n))}{n} \right| \leq \ln n \cdot e^{(\sqrt{2} + \epsilon) \sqrt{\ln \ln n}}.$$  

(11)
Because \(\lim_{n \to \infty} a_n = 0\), the equation \(\lim_{n \to \infty} \ln n \cdot e^{-\left(\sqrt{2 + a_n}\right) \sqrt{\ln \ln n}} = 0\) is found true, thus

\[
\liminf_{n \to \infty} \frac{n}{\ln n} \cdot E(P(n)) = \liminf_{n \to \infty} \frac{E(S(n))}{\ln n} = \limsup_{n \to \infty} \frac{E(P(n))}{n} \quad \text{holds.}
\]

Theorem 2 is obtained as a direct consequence of Theorem 1.

Theorem 2

\[E(P(n)) = O\left(\frac{n}{\ln n}\right)\]

Proof The equation \(\frac{1}{2} \leq \liminf_{n \to \infty} \frac{n}{\ln n} \cdot E(P(n)) \leq \limsup_{n \to \infty} \frac{n}{\ln n} \cdot E(P(n)) \leq 1\) is found true applying Theorem 1. From that, there is a natural number \(N\), such that that

\[
(\forall n \geq N) \left(\frac{1}{2} - \varepsilon\right) \cdot \frac{n}{\ln n} \leq E(P(n)) \leq (1 + \varepsilon) \cdot \frac{n}{\ln n}.
\]

Therefore, the equation \(E(P(n)) = O\left(\frac{n}{\ln n}\right)\) holds.

The right question that comes from (12) is the following "Is the equation \(E(P(n)) \leq \frac{n}{\ln n}\) true?". This has been investigated for all the natural numbers less than 1000000 and it has been found true. Equation (7) can be adapt to the average \(E(P(n))\) but obviously the inequality that is found is not an answer to the question. Therefore, we may conjecture the following: The equation \(E(P(n)) \leq \frac{n}{\ln n}\) holds for all \(n > 1\).

References


ON THE CONVERGENCE OF THE ERDOS HARMONIC SERIES

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The purpose of this article is to study the convergence of a few series with the Erdos function. The work is based on results concerning the convergence of some series with the Smarandache function.

1. INTRODUCTION

The results used in this article are presented briefly in the following. These concern the relationship between the Smarandache and the Erdos functions and the convergence of some series. These two functions are important function in Number Theory. They are defined as follows:

- The Smarandache function [Smarandache, 1980] is \( S : \mathbb{N}^* \rightarrow \mathbb{N} \) defined by

\[
S(n) = \min\{k \in \mathbb{N} | k! = M_n \} \quad (\forall n \in \mathbb{N}^* ).
\]

- The Erdos function is \( P : \mathbb{N}^* \rightarrow \mathbb{N} \) defined by

\[
P(n) = \min\{p \in \mathbb{N} | n = Mp \land p \text{ is prim} \} \quad (\forall n \in \mathbb{N}^* \setminus \{1\}), \quad P(1) = 0 .
\]

The main properties of them are:

\[
(\forall a, b \in \mathbb{N}^*) \quad (a, b) = 1 \Rightarrow S(a \cdot b) = \max\{S(a), S(b)\}, \quad P(a \cdot b) = \max\{P(a), P(b)\}.
\]

\[
(\forall a \in \mathbb{N}^*) \quad P(a) \leq S(a) \leq a \quad \text{and the equalities occur iif a is prim.}
\]

Erdos [1991] found the relationship between these two functions that is given by

\[
\lim_{n \to \infty} \left| \left\{ \frac{\varphi = 1, n \mid P(i) < S(i) \right\} \right| = 0.
\]

This important result was extended by Ford [1999] to

\[
\left| \left\{ \varphi = 1, n \mid P(i) < S(i) \right\} \right| = n \cdot e^{-\left(\frac{1}{2} + a_n\right) - a_n a_n a_n}, \quad \text{where } \lim_{n \to \infty} a_n = 0.
\]

Equations (5-6) are very important because allow us to translate convergence properties on the Smarandache function to convergence properties on the Erdos function. This translation represents the main technique that is used to obtain the convergence of some series with the function \( P \).

2. THE ERDOS HARMONIC SERIES
The Erdos harmonic series can be defined by \( \sum_{n \geq 2} \frac{1}{P^a(n)} \). This is one of the important series with the Erdos function and its convergence is studied starting from the convergence of the Smarandache harmonic series \( \sum_{n \geq 2} \frac{1}{S^a(n)} \). Some results concerning series with the function \( S \) are reviewed briefly in the following:

- If \((x_n)_{n \geq 0}\) is an increasing sequence such that \( \lim_{n \to \infty} x_n = \infty \), then the series \( \sum_{n \geq 1} \frac{x_{n+1} - x_n}{S(x_n)} \) is divergent. [Cojocaru, 1997].

\( \text{(7)} \)

- The series \( \sum_{n \geq 2} \frac{1}{S^2(n)} \) is divergent. [Tabirca, 1998]

\( \text{(8)} \)

- The series \( \sum_{n \geq 2} \frac{1}{S^a(n)} \) is divergent for all \( a > 0 \). [Luca, 1999]

\( \text{(9)} \)

These above results are translated to the similar properties on the Erdos function.

**Theorem 1.** If \((x_n)_{n \geq 0}\) is an increasing sequence such that \( \lim_{n \to \infty} x_n = \infty \), then the series \( \sum_{n \geq 1} \frac{x_{n+1} - x_n}{P(x_n)} \) is divergent.

**Proof** The proof is obvious based on the equation \( P(x_n) \leq S(x_n) \). Therefore, the equation \( \frac{x_{n+1} - x_n}{P(x_n)} \geq \frac{x_{n+1} - x_n}{S(x_n)} \) and the divergence of the series \( \sum_{n \geq 1} \frac{x_{n+1} - x_n}{S(x_n)} \) give that the series

\( \sum_{n \geq 1} \frac{x_{n+1} - x_n}{P(x_n)} \) is divergent.

\*
A direct consequence of Theorem 1 is the divergence of the series \( \sum_{n \geq 1} \frac{1}{P(a \cdot n + b)} \), where \( a, b > 0 \) are positive numbers. This gives that \( \sum_{n \geq 1} \frac{1}{P(n)} \) is divergent and moreover that \( \sum_{n \geq 2} \frac{1}{P^a(n)} \) is divergent for all \( a < 1 \).

**Theorem 2.** The series \( \sum_{n \geq 2} \frac{1}{P^a(n)} \) is divergent for all \( a > 1 \).

**Proof.** The proof studies two cases.

**Case 1.** \( a \geq \frac{1}{2} \).

In this case, the proof is made by using the divergence of \( \sum_{n \geq 2} \frac{1}{S^a(n)} \).

Denote \( A = \{ i = 2, n \mid S(i) = P(i) \} \) and \( B = \{ i = 2, n \mid S(i) > P(i) \} \) a partition of the set \( \{ i = 1, n \} \). We start from the following simple transformation

\[
\sum_{i=2}^{n} \frac{1}{P^a(i)} = \sum_{i=2}^{n} \frac{1}{S^a(i)} + \sum_{i \in B} \left[ \frac{1}{P^a(i)} - \frac{1}{S^a(i)} \right] = \sum_{i=2}^{n} \frac{1}{S^a(i)} + \sum_{i \in B} \frac{S^a(i) - P^a(i)}{P^a(i) \cdot S^a(i)} .
\]

An \( i \in B \) satisfies \( S^a(i) - P^a(i) \geq 1 \) and \( P(i) < S(i) \leq n \) thus, (10) becomes

\[
\sum_{i=2}^{n} \frac{1}{P^a(i)} \geq \sum_{i=2}^{n} \frac{1}{S^a(i)} + \sum_{i \in B} \frac{1}{n^2} = \sum_{i=2}^{n} \frac{1}{S^a(i)} + \frac{1}{n^2} \cdot |B| .
\]

The series \( \sum_{n \geq 2} \frac{1}{P^a(n)} \) is divergent because the series \( \sum_{n \geq 2} \frac{1}{S^a(n)} \) is divergent and

\[
\lim_{n \to \infty} \frac{|B|}{n^2} = \lim_{n \to \infty} \frac{n \cdot e^{(\sqrt{2} + a) \sqrt{\ln n \cdot \ln n}}}{n^2} = \lim_{n \to \infty} \frac{1}{n^{2a-1} \cdot e^{(\sqrt{2} + a) \sqrt{\ln n \cdot \ln n}}} = 0 .
\]

**Case 2.** \( \frac{1}{2} > a > 1 \).

The first case gives that the series \( \sum_{n \geq 2} \frac{1}{P^a(n)} \) is divergent.
Based on $P^2(n) > P^a(n)$, the inequality $\sum_{i=2}^{n} \frac{1}{P^a(i)} > \sum_{i=2}^{n} \frac{1}{P^2(i)}$ is found. Thus, the series

$$\sum_{n=2}^{\infty} \frac{1}{S^a(n)}$$

is divergent.

The technique that has been applied to the proof of Theorem 2 can be used in both ways. Theorem 2 started from a property of the Smarandache function and found a property of the Erdos function. Opposite, Finch [1999] found the property $\lim_{n \to \infty} \frac{\sum_{i=2}^{n} \ln S(i)}{\ln n} = \lambda$ based on the similar property $\lim_{n \to \infty} \frac{\sum_{i=2}^{n} \ln P(i)}{\ln n} = \lambda$, where $\lambda = 0.6243299$ is the Golomb-Dickman constant. Obviously, many other properties can be proved using this technique. Moreover, Equations (5-6) gives a very interesting fact - "the Smarandache and Erdos function may have the same behavior on the convergence problems."

References


An experimental evidence on the validity of third Smarandache conjecture on primes

Felice Russo

Abstract

In this note we report the results regarding the check of the third Smarandache conjecture on primes [1],[2] for $p_n \leq 2^{25}$ and $2 \leq k \leq 10$. In the range analysed the conjecture is true. Moreover, according to experimental data obtained, it seems likely that the conjecture is true for all primes and for all positive values of $k$.

Introduction

In [1] and [2] the following function has been defined:

$$C(n, k) = \frac{1}{p_{n+k}} - \frac{1}{p_n}$$

where $p_n$ is the n-th prime and $k$ is a positive integer. Moreover in the above mentioned papers the following conjecture has been formulated by F. Smarandache:

$$C(n, k) < \frac{2}{k} \text{ for } k \geq 2$$

This conjecture is the generalization of the Andrica conjecture ($k=2$) [3] that has not yet been proven. This third Smarandache conjecture has been tested for $p_n \leq 2^{25}$, $2 \leq k \leq 10$ and in this note the result of this search is reported. The computer code has been written utilizing the Ubasic software package.

Experimental Results

In the following graph the Smarandache function for $k=4$ and $n \leq 1000$ is reported. As we can see the value of $C(k, n)$ is modulated by the prime's gap indicated by $d_n = p_{n+1} - p_n$. We call this graph the Smarandache "comet".
In the following table, instead, we report:

- the largest value $\text{Max}_k C(n,k)$ of Smarandache function for $2 \leq k \leq 10$ and $p_n \leq 2^{25}$
- the difference $\Delta(k)$ between $2/k$ and $\text{Max}_k C(n,k)$
- the value of $p_n$ that maximize $C(n,k)$
- the value of $2/k$

<table>
<thead>
<tr>
<th>$k$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
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</thead>
<tbody>
<tr>
<td>Max $C(n,k)$</td>
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<tr>
<td>$\Delta$</td>
<td>0.32913</td>
<td>0.35562</td>
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<td>0.26038</td>
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<td>0.19715</td>
<td>0.17436</td>
<td>0.15624</td>
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<td>$p_n$</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>3</td>
<td>3</td>
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<td>3</td>
</tr>
<tr>
<td>$2/k$</td>
<td>1</td>
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<td>0.5</td>
<td>0.4</td>
<td>0.333..</td>
<td>0.2857..</td>
<td>0.25</td>
<td>0.222..</td>
<td>0.20</td>
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</table>

According to previous data the third Smarandache conjecture is verified in the range of $k$ and $p_n$ analysed due to the fact that $\Delta$ is always positive. Moreover since the Smarandache $C(n,k)$ function falls asymptotically as $n$ increases it is likely that the estimated maximum is valid also for $p_n > 2^{25}$. 
We can also analyse the behaviour of difference $\Delta(k)$ versus the k parameter that in the following graph is showed with white dots. We have estimated an interpolating function:

$$\Delta(k) \approx 0.88 \cdot \frac{1}{k^{0.78}} \text{ for } k > 2$$

with a very good $R^2$ value (see the continuous curve). This result reinforces the validity of the third Smarandache conjecture since:

$$\Delta(k) \to 0 \text{ for } k \to \infty$$

New Question

According to previous experimental data can we reformulate the third Smarandache conjecture with a tighter limit as showed below?

$$C(n,k) < \frac{2}{2 \cdot a_0^k}$$

where $k \geq 2$ and $a_0$ is the Smarandache constant [4],[1]
References:


Some results about four Smarandache U-product sequences

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Abstract

In this paper four Smarandache product sequences have been studied: Smarandache Square product sequence, Smarandache Cubic product sequence, Smarandache Factorial product sequence and Smarandache Palprime product sequence. In particular the number of primes, the convergence value for Smarandache Series, Smarandache Continued Fractions, Smarandache Infinite product of the mentioned sequences has been calculated utilizing the Ubasic software package. Moreover for the first time the notion of Smarandache Continued Radicals has been introduced. One conjecture about the number of primes contained in these sequences and new questions are posed too.

Introduction

In [1] Iacobescu describes the so called Smarandache U-product sequence. Let \( u, \ n \geq 1 \), be a positive integer sequence. Then a U-sequence is defined as follows:

\[
U_n = 1 + u_1 \cdot u_2 \cdot \ldots \cdot u_n
\]

In this paper differently from [1], we will call this sequence a U-sequence of the first kind because we will introduce for the first time a U-sequence of the second kind defined as follows:

\[
U_n = \left| 1 - u_1 \cdot u_2 \cdot \ldots \cdot u_n \right|
\]

In this paper we will discuss about the “Square product”, “Cubic product”, “Factorial product” and “Primorial product” sequences. In particular we will analyze the question posed by Iacobescu in [1] on the number of primes contained in those sequences. We will also analyze the convergence values of the Smarandache Series [2], Infinite product [3], Simple Continued Fractions [4] of the four sequences. Moreover for the first time we will introduce the notion of Smarandache Continued Radicals and we will analyze the convergence of sequences reported above.
Sequences details

- **Smarandache square product sequence of the first and second kind.**

In this case the sequence \( u_n \) is given by:

\[
1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, \ldots.
\]

that is the square of \( n \). The first 20 terms of the sequence \( U_n \) \((1 \leq n \leq 20)\) both the first and second kind are reported in the table below:

<table>
<thead>
<tr>
<th>Smarandache Square product sequence (first kind)</th>
<th>Smarandache Square product sequence (second kind)</th>
</tr>
</thead>
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</tr>
</tbody>
</table>

- **Smarandache cubic product sequence of the first and second kind.**

In this case the sequence \( u_n \) is given by:

\[
1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728, \ldots.
\]

that is the cube of \( n \). Here the first 17 terms for the sequence \( U_n \) of the first and second kind.

43
<table>
<thead>
<tr>
<th>Smarandache Cubic product sequence (first kind)</th>
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</tr>
<tr>
<td>4778472583987200001</td>
<td>4778472583987199999</td>
</tr>
</tbody>
</table>

\[ o \text{ Smarandache factorial product sequence of the first and second kind.} \]

In this case the sequence \( u_n \) is given by:

\[ 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \ldots \]

that is the factorial of \( n \). The first 13 terms of the \( U_n \) sequence of the first and second kind follow.

<table>
<thead>
<tr>
<th>Smarandache Factorial product sequence (first kind)</th>
<th>Smarandache Factorial product sequence (second kind)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>259</td>
<td>11</td>
</tr>
<tr>
<td>34561</td>
<td>287</td>
</tr>
<tr>
<td>34883201</td>
<td>34559</td>
</tr>
<tr>
<td>125411338001</td>
<td>125411327999</td>
</tr>
<tr>
<td>505658474960001</td>
<td>505658474959999</td>
</tr>
<tr>
<td>1834933472251084800001</td>
<td>183493347225108479999</td>
</tr>
<tr>
<td>66586066584010476522400000001</td>
<td>66586066584010476522239999999</td>
</tr>
<tr>
<td>26790267296391946810949632000000001</td>
<td>26790267296391946810949631999999999</td>
</tr>
<tr>
<td>12731396329909416749597712474112000000000001</td>
<td>1273139632990941674959771247411199999999999</td>
</tr>
<tr>
<td>7927866975957679560737708640087148852960000000000001</td>
<td>7927866975957679560737708640087148852959999999999999999</td>
</tr>
</tbody>
</table>

44
In this case the sequence \( u_n \) is given by:

\[
2, 3, 5, 7, 11, 121, 131, 151, 181, 191, 313, 353, 353, 373, \ldots
\]

that is the sequence of palindromic primes. Below the first 17 terms of the \( U_n \) sequence of the first and second kind.

<table>
<thead>
<tr>
<th>Smarandache Palprime product sequence (first kind)</th>
<th>Smarandache Palprime product sequence (second kind)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>31</td>
<td>29</td>
</tr>
<tr>
<td>211</td>
<td>209</td>
</tr>
<tr>
<td>2311</td>
<td>2309</td>
</tr>
<tr>
<td>23311</td>
<td>233309</td>
</tr>
<tr>
<td>28230511</td>
<td>28230509</td>
</tr>
<tr>
<td>3698196811</td>
<td>3698196809</td>
</tr>
<tr>
<td>558427718311</td>
<td>558427718309</td>
</tr>
<tr>
<td>101075417014111</td>
<td>101075417014109</td>
</tr>
<tr>
<td>1930540469695011</td>
<td>1930540469695009</td>
</tr>
<tr>
<td>6042591655534538131</td>
<td>6042591655534538129</td>
</tr>
<tr>
<td>2133034854340151959891</td>
<td>2133034854340151959889</td>
</tr>
<tr>
<td>7956220066876681038971</td>
<td>7956220066876681038969</td>
</tr>
<tr>
<td>304723226256179768837925511</td>
<td>304723226256179768837925509</td>
</tr>
<tr>
<td>221533785488242691945171845771</td>
<td>221533785488242691945171845769</td>
</tr>
<tr>
<td>167701075614599717802495087247891</td>
<td>167701075614599717802495087247889</td>
</tr>
</tbody>
</table>

Results

For all above sequences the following questions have been studied:

1. How many terms are prime?
2. Is the Smarandache Series convergent?
3. Is the Smarandache Infinite product convergent?
4. Is the Smarandache Simple Continued Fractions convergent?
5. Is the Smarandache Continued Radicals convergent?

For this purpose the software package Ubasic Rev. 9 has been utilized. In particular for the item n. 1, a strong pseudoprime test code has been written [5]. Moreover, as already mentioned above, the item 5 has been introduced for the first time; a Smarandache Continued Radicals is defined as follows:

\[
\sqrt{a(1)} + \sqrt{a(2)} + \sqrt{a(3)} + \sqrt{a(4)} + \ldots
\]
where \( a(n) \) is the \( n \)th term of a Smarandache sequence. Here below a summary table of the obtained results:

<table>
<thead>
<tr>
<th></th>
<th># Primes</th>
<th>SS cv</th>
<th>SIP cv</th>
<th>SSCF cv</th>
<th>SCR cv</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square 1(^{st}) kind</td>
<td>12456=0.026</td>
<td>0.7288315379...</td>
<td>2.1989247812...</td>
<td>2.3666079803...</td>
<td></td>
</tr>
<tr>
<td>Square 2(^{nd}) kind</td>
<td>1463=0.0021</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>0.3301888340...</td>
<td>1.8143775546...</td>
</tr>
<tr>
<td>Cubic 1(^{st}) kind</td>
<td>( \infty )</td>
<td>0.6157923201...</td>
<td>2.1110542477...</td>
<td>2.6904314681...</td>
<td></td>
</tr>
<tr>
<td>Cubic 2(^{nd}) kind</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>0.1427622842...</td>
<td>2.244613806...</td>
<td></td>
</tr>
<tr>
<td>Factorial 1(^{st}) kind</td>
<td>570=0.071</td>
<td>0.9137455924...</td>
<td>2.3250021620...</td>
<td>2.2332152218...</td>
<td></td>
</tr>
<tr>
<td>Factorial 2(^{nd}) kind</td>
<td>266=0.033</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>0.9166908563...</td>
<td>1.6117607295...</td>
</tr>
<tr>
<td>Palprime 1(^{st}) kind</td>
<td>10363=0.027</td>
<td>0.5136249121...</td>
<td>3.1422019345...</td>
<td>2.5932060878...</td>
<td></td>
</tr>
<tr>
<td>Palprime 2(^{nd}) kind</td>
<td>9363=0.024</td>
<td>1.2397048573...</td>
<td>1.1986303614...</td>
<td>2.1032632883...</td>
<td></td>
</tr>
</tbody>
</table>

Legend:

- \( \# \) primes: (Number of primes/number of sequence terms checked)
- SS cv: (Smarandache Series convergence value)
- SIP cv: (Smarandache Infinite Product convergence value)
- SSCF cv: (Smarandache Simple Continued Fractions convergence value)
- SCR cv: (Smarandache Continued Radicals convergence value)
- @: (This sequence contain only one prime as proved by M. Le and K. Wu [6])

About the items 2,3,4 and 5 according to these results the answer is: yes, all the analyzed sequences converge except the Smarandache Series and the Smarandache Infinite product for the square product (2\(^{nd}\) kind), cubic product (2\(^{nd}\) kind) and factorial product (2\(^{nd}\) kind). In particular notice the nice result obtained with the convergence of Smarandache Simple Continued Fractions of Smarandache palprime product sequence of the first kind.

The value of convergence is roughly \( \pi \) with the first two decimal digits correct.

\[
\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{31 + \cfrac{1}{211 + \cfrac{1}{2311 + \cfrac{1}{233311 + \ddots}}}}}
\]

Analogously for the cubic product sequence of the second kind the simple continued fraction converge roughly to \( \pi - 3 \), while for the factorial product sequence of the second kind the continued radical converge roughly (two first decimal digits correct) to the golden ratio \( \phi \), that is:

\[
\phi \approx 1.618033989
\]
About the item 1, the following table reports the values of n in the sequence that generate a strong pseudoprime number and its digit's number.

<table>
<thead>
<tr>
<th>n</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square 1&lt;sup&gt;st&lt;/sup&gt; kind</td>
<td>1/2/3/4/5/9/10/11/1324/65/76</td>
</tr>
<tr>
<td>Square 2&lt;sup&gt;nd&lt;/sup&gt; kind</td>
<td>2</td>
</tr>
<tr>
<td>Cubic 1&lt;sup&gt;st&lt;/sup&gt; kind</td>
<td>1</td>
</tr>
<tr>
<td>Cubic 2&lt;sup&gt;nd&lt;/sup&gt; kind</td>
<td>2</td>
</tr>
<tr>
<td>Factorial 1&lt;sup&gt;st&lt;/sup&gt; kind</td>
<td>1/2/3/7/14</td>
</tr>
<tr>
<td>Factorial 2&lt;sup&gt;nd&lt;/sup&gt; kind</td>
<td>3/7</td>
</tr>
<tr>
<td>Palprime 1&lt;sup&gt;st&lt;/sup&gt; kind</td>
<td>1/2/3/4/5/7/10/19/57/234</td>
</tr>
<tr>
<td>Palprime 2&lt;sup&gt;nd&lt;/sup&gt; kind</td>
<td>2/3/4/5/7/10/19/57/234</td>
</tr>
</tbody>
</table>

Please note that the primes in the sequence of palprime of the first and second kind generate pairs of twin primes. The first ones follow:

(3,5) (5,7) (29,31) (209,211) (2309,2311) (28230509,28230511) (101075417014109,101075417014111) .......

Due to the fact that the percentage of primes found is very small and that according to Prime Number Theorem, the probability that a randomly chosen number of size n is prime decreases as 1/d (where d is the number of digits of n) we are enough confident to pose the following conjecture:

- The number of primes contained in the Smarandache Square product sequence (1<sup>st</sup> and 2<sup>nd</sup> kind), Smarandache Factorial product sequence (1<sup>st</sup> and 2<sup>nd</sup> kind) and Smarandache Palprime product sequence (1<sup>st</sup> and 2<sup>nd</sup> kind) is finite.
New Questions

- Is there any Smarandache sequence whose SS, SIP, SSCF and SCR converge to some known mathematical constants?

- Are all the estimated convergence values irrational or transcendental?

- Is there for each prime inside the Smarandache Palprime product sequence of the second kind the correspondent twin prime in the Smarandache Palprime product sequence of the first kind?

- Are there any two Smarandache sequences $a(n)$ and $b(n)$ whose Smarandache Infinite Product ratio converge to some value $k$ different from zero?

$$\lim_{n \to \infty} \prod_{n} \frac{1}{a(n)} = k$$

$$\lim_{n \to \infty} \prod_{n} \frac{1}{b(n)}$$

- Is there any Smarandache sequence $a(n)$ such that:

$$\lim_{n \to \infty} e^{\sum_{n} \frac{1}{a(n)}} \approx \pi$$

- For the four sequences of first kind $a(n)$, study:

$$\lim_{n \to \infty} \sum_{n} \frac{a(n)}{R(a(n))}$$

where $R(a(n))$ is the reverse of $a(n)$. (For example if $a(n)=17$ then $R(a(n))=71$ and so on).
References

On an unsolved question about the Smarandache Square-Partial-Digital Subsequence

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Abstract

In this note we report the solution of an unsolved question on Smarandache Square-Partial-Digital Subsequence. We have found it by extensive computer search. Some new questions about palindromic numbers and prime numbers in SSPDS are posed too.

Introduction

The Smarandache Square-Partial-Digital Subsequence (SSPDS) is the sequence of square integers which admits a partition for which each segment is a square integer [1],[2],[3]. The first terms of the sequence follow:

49, 144, 169, 361, 441, 1225, 1369, 1444, 1681, 1936, 3249, 4225, 4900, 11449, 12544, 14641, 15625, 16900 ...

or

7, 12, 13, 19, 21, 35, 37, 38, 41, 44, 57, 65, 70, 107, 112, 121, 125, 130, 190, 191, 204, 205, 209, 212, 223, 253 ...

reporting the value of n^2 that can be partitioned into two or more numbers that are also squares (A048653) [5]. Differently from the sequences reported in [1],[2] and [3] the proposed ones don’t contain terms that admit 0 as partition. In fact as reported in [4] we don’t consider the number zero a perfect square. So, for example, the term 256036 and the term 506 respectively, are not reported in the above sequences because the partition 2560/36 contains the number zero.

L. Widmer explored some properties of SSPDS’s and posed the following question [2]:

Is there a sequence of three or more consecutive integers whose squares are in SPDS?

This note gives an answer to this question.

Results

A computer code has been written in Ubasic Rev. 9. After about three week of work only a solution for three consecutive integers has been found. Those consecutive integers are: 12225, 12226, 12227.
No other three consecutive integers or more have been found for terms in SSPDS up to about $3.3 \times 10^9$. Below a graph of distance $d_n$ between the terms of sequence A048653 versus $n$ is given; in particular $d_n = a(n+1) - a(n)$ where $n$ is the $n$-th term of the sequence.

![Graph of distance $d_n$ between terms of sequence A048653 versus $n$.]

According to the previous results we are enough confident to offer the following conjecture:

- **There are no four consecutive integers whose squares are in SSPDS.**

**New Questions**

Starting with the sequence (A048646), reported above, the following sequence can be created [5] (A048653):

7, 13, 19, 37, 41, 107, 191, 223, 379, 487, 1093, 1201, 1301, 1907, 3019, 3371, 5081, 9041, 9721, 9907......

that we can call "Smarandache Prime-Square-Partial-Digital-Subsequence" because all the squares of these primes can be partitioned into two or more numbers that are also squares.

By looking this sequence the following questions can be posed:

1. Are there other palindromic primes in this sequence beyond the palprime 191?
2. Is there at least one palindromic prime in this sequence which square is a palindromic square?
3. Are there in this sequence other two or more consecutive primes beyond 37 and 41?
If we look now at the terms of the sequence A048653 we discover that two of them are very interesting:

121 and 212

Both numbers are palindromes and their squares are in SSPDS and palindromes too. In fact $121^2 = 14641$ can be partitioned as: $1, 4, 6, 4, 1$ and $212^2 = 44944$ can be partitioned in five squares that are also palindromes: $4, 4, 9, 4, 4$. These are the only terms found by our computer search. So the following question arises:

1. **How many other SSPDS palindromes do exist?**

References

The Smarandache Square-Partial-Digital Subsequence (SSPDS) is the sequence of square integers which can be partitioned so that each element of the partition is a perfect square\(^2\). For example, 3249 is in SSPDS since 3249 can be partitioned into \(324 = 18^2\) and \(9 = 3^2\).

The first terms of the sequence are:

49, 144, 169, 361, 441, 1225, 1369, 1444, 1681, 1936, 3249, 4225, 4900, 11449, 12544, 14641, ...

where the square roots are

7, 12, 13, 19, 21, 35, 37, 38, 41, 44, 57, 65, 70, 107, 112, 121, ...

this sequence is assigned the identification code A048653[4].

L. Widmer examined this sequence and posed the following question[2]:

Is there a sequence of three or more consecutive integers whose squares are in SPDS?

For the purposes of this examination, we will assume that 0 is not a perfect square. For example, the number 90 will not be considered as a number that can be partitioned into two perfect squares. Furthermore, elements of the partition are not allowed to have leading zeros. For example, 101 cannot be partitioned into perfect squares.

Russo[5] considered this question and concluded that the only additional solution to the Widmer question up to \(3.3 \times 10^9\) was

\[
\begin{array}{ccc}
\text{n} & n^2 & \text{Partition} \\
12225 & 149450625 & 1,4,9,4,50625 \\
12226 & 149475076 & 1,4,9,4,75076 \\
12227 & 149499529 & 1,4,9,4,9,9,529
\end{array}
\]

and made the following conjecture:

There are no four consecutive integers whose squares are in SSPDS.

The purpose of this short paper is to present several additional solutions to the Widmer question as well as a counterexample to the Russo conjecture.

A computer program was written in the language Delphi Ver. 4 and run for all numbers \(n\), where \(n \leq 100,000,000\) and the following ten additional solutions were found

\[
\begin{array}{ccc}
\text{n} & n^2 & \text{Partition} \\
376779 & 141962414841 & 1, 4, 1, 9, 6241, 4, 841 \\
376780 & 141963168400 & 1, 4, 196, 3168400 \\
376781 & 141963921961 & 1, 4, 196392196, 1
\end{array}
\]
<table>
<thead>
<tr>
<th>n</th>
<th>n^2</th>
<th>Partition</th>
</tr>
</thead>
<tbody>
<tr>
<td>974379</td>
<td>949414435641</td>
<td>9, 4, 9, 4, 1, 4, 4356, 4, 1</td>
</tr>
<tr>
<td>974380</td>
<td>949416384400</td>
<td>949416384400</td>
</tr>
<tr>
<td>974381</td>
<td>949418333161</td>
<td>9, 4, 9, 9, 1833316, 1</td>
</tr>
<tr>
<td>999055</td>
<td>998110893025</td>
<td>9, 9, 81, 1089, 3025</td>
</tr>
<tr>
<td>999056</td>
<td>998112891136</td>
<td>9, 9, 81, 1, 289, 1, 1, 36</td>
</tr>
<tr>
<td>999057</td>
<td>998114889249</td>
<td>9, 9, 81, 1, 4, 889249</td>
</tr>
<tr>
<td>999058</td>
<td>998112891136</td>
<td>9, 9, 81, 1, 289, 1, 1, 36</td>
</tr>
<tr>
<td>999057</td>
<td>998114889249</td>
<td>9, 9, 81, 1, 4, 889249</td>
</tr>
<tr>
<td>2000341</td>
<td>4001364116281</td>
<td>400, 1, 36, 4, 116281</td>
</tr>
<tr>
<td>2000342</td>
<td>4001368116964</td>
<td>400, 1, 36, 81, 16, 9, 64</td>
</tr>
<tr>
<td>2000343</td>
<td>4001372117649</td>
<td>400, 13721, 1764, 9</td>
</tr>
<tr>
<td>2063955</td>
<td>4259910242025</td>
<td>4, 25, 9, 9, 1024, 2025</td>
</tr>
<tr>
<td>2063956</td>
<td>4259914369936</td>
<td>4, 25, 9, 9, 14, 36, 9, 9, 36</td>
</tr>
<tr>
<td>2063957</td>
<td>4259918497849</td>
<td>4, 25, 9, 9, 1849, 784, 9</td>
</tr>
<tr>
<td>2083941</td>
<td>4342810091481</td>
<td>4342810091481</td>
</tr>
<tr>
<td>2083942</td>
<td>4342814259364</td>
<td>4342814259364</td>
</tr>
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<td>2083943</td>
<td>4342818427249</td>
<td>4342818427249</td>
</tr>
<tr>
<td>4700204</td>
<td>22091917641616</td>
<td>2209, 1, 9, 1764, 16, 16</td>
</tr>
<tr>
<td>4700205</td>
<td>22091927042025</td>
<td>2209, 1, 9, 2704, 2025</td>
</tr>
<tr>
<td>4700206</td>
<td>22091936442436</td>
<td>2209, 1, 9, 36, 4, 42436</td>
</tr>
<tr>
<td>5500374</td>
<td>30254114139876</td>
<td>3025, 4, 1, 1, 4, 139876</td>
</tr>
<tr>
<td>5500375</td>
<td>30254125140625</td>
<td>3025, 4, 1, 25, 140625</td>
</tr>
<tr>
<td>5500376</td>
<td>30254136141376</td>
<td>3025, 4, 1, 36, 141376</td>
</tr>
<tr>
<td>80001024</td>
<td>6400163841048576</td>
<td>6400163841048576</td>
</tr>
<tr>
<td>80001025</td>
<td>6400164001050625</td>
<td>6400164001050625</td>
</tr>
<tr>
<td>80001026</td>
<td>6400164161052676</td>
<td>6400164161052676</td>
</tr>
</tbody>
</table>
Pay particular attention to the four consecutive numbers 999055, 999056, 999057 and 999058. These four numbers are a counterexample to the conjecture by Russo.

Given the frequency of three consecutive integers whose squares are in SSPDS, the following conjecture is made:

There are an infinite number of three consecutive integer sequences whose squares are in SSPDS.

In terms of larger sequences, the following conjecture also appears to be a safe one:

There is an upper limit to the length of consecutive integer sequences whose squares are in SSPDS.

We close with an unsolved question:

What is the length of the largest sequence of consecutive integers whose squares are in SSPDS?

References


A functional recurrence to obtain the prime numbers using the Smarandache prime function.

Sebastián Martín Ruiz. Avda de Regla, 43. Chipiona 11550 Cádiz Spain.

Theorem: We are considering the function:

For $n \geq 2$, integer:

$$F(n) = n + 1 + \sum_{m=n+1}^{2n} \prod_{m=n+1}^{m} \left[ -E \left( \frac{\sum_{i=1}^{m} (\lfloor \frac{i}{p_k} \rfloor - \lfloor \frac{i}{p_{k-1}} \rfloor )} {\sum_{i=1}^{m} (\lfloor \frac{i}{p_k} \rfloor - \lfloor \frac{i}{p_{k-1}} \rfloor )} \right) \right]$$

one has: $p_{k+1} = F(p_k)$ for all $k \geq 1$ where $\{p_k\}_{k \geq 1}$ are the prime numbers and $E(x)$ is the greatest integer less than or equal to $x$.

Observe that the knowledge of $p_{k+1}$ only depends on knowledge of $p_k$ and the knowledge of the fore primes is unnecessary.

Observe that this is a functional recurrence strictly closed too.

Proof:

Suppose that we have found a function $G(i)$ with the following property:

$$G(i) = \begin{cases} 1 & \text{if } i \text{ is compound} \\ 0 & \text{if } i \text{ is prime} \end{cases}$$

This function is called Smarandache Prime Function (Reference)

Consider the following product:

$$\prod_{i=p_k+1}^{m} G(i)$$

If $p_k < m < p_{k+1}$, $\prod_{i=p_k+1}^{m} G(i) = 1$ since $i : p_k + 1 \leq i \leq m$ are all compounds.
If \( m \geq p_{k+1} \) \( \prod_{m=p_{k+1}}^{m} G(i) = 0 \) since the \( G(p_{k+1}) = 0 \) factor is in the product.

Here is the sum:

\[
\sum_{m=p_{k+1}}^{2p_{k}} \prod_{m=p_{k+1}}^{m} G(i) = \sum_{m=p_{k+1}}^{p_{k+1}-1} G(i) + \sum_{m=p_{k+1}}^{2p_{k}} \prod_{m=p_{k+1}}^{m} G(i) = \sum_{m=p_{k+1}}^{p_{k+1}-1} 1 =
\]

\( = p_{k+1} - 1 - (p_{k} + 1) + 1 = p_{k+1} - p_{k} \)

The second sum is zero since all products have the factor \( G(p_{k+1}) = 0 \).

Therefore we have the following relation of recurrence:

\[
p_{k+1} = p_{k} + 1 + \sum_{m=p_{k+1}}^{2p_{k}} \prod_{m=p_{k+1}}^{m} G(i)
\]

Let’s now see that we can find \( G(i) \) with the asked property.

Considerer:

(1) \( E\left( \frac{i}{j} \right) - E\left( \frac{i-1}{j} \right) = \begin{cases} 
1 & \text{if } j \mid i \text{ and } j = 1, 2, \ldots, i \\
0 & \text{else}
\end{cases} \quad \text{for } i \geq 1
\]

We shall deduce this later.

We deduce of this relation:

\[
d(i) = \sum_{m=1}^{i} \left( E\left( \frac{i}{j} \right) - E\left( \frac{i-1}{j} \right) \right) \quad \text{where } d(i) \text{ is the number of divisors of } i.
\]

If \( i \) is prime \( d(i) = 2 \) therefore:

\[
-E\left[ \frac{d(i-2)}{d(i-1)} \right] = 0
\]

If \( i \) is compound \( d(i) > 2 \) therefore:

\[
0 < \frac{d(i-2)}{d(i-1)} < 1 \Rightarrow -E\left[ \frac{d(i-2)}{d(i-1)} \right] = 1
\]
Therefore we have obtained the function $G(i)$ which is:

$$G(i) = -E \left[ \frac{\sum_{j=1}^{i}(E(\frac{1}{j}) - E(\frac{m+1}{j})) - 1}{\sum_{j=1}^{i}(E(\frac{1}{j}) - E(\frac{m+1}{j})) - 1} \right]$$

$i \geq 2$ integer

To finish the demonstration of the theorem it is necessary to prove (1)

If $j = 1$  

$$j \mid i \quad E(\frac{1}{j}) - E(\frac{m+1}{j}) = i - (i - 1) = 1$$

If $j > 1$

$$i = jE(\frac{1}{j}) + r \quad 0 \leq r < j$$
$$i - 1 = jE(\frac{m+1}{j}) + s \quad 0 \leq s < j$$

If $j \mid i \Rightarrow r = 0 \Rightarrow jE(\frac{1}{j}) = jE(\frac{m+1}{j}) + s + 1 \Rightarrow j \mid s + 1$$
$$s + 1 \leq j \Rightarrow j = s + 1$$

$$= jE(\frac{1}{j}) = jE(\frac{m+1}{j})$$
$$+ j \Rightarrow E(\frac{1}{j}) = E(\frac{m+1}{j})$$

If $j \not\mid i \Rightarrow r > 0 \Rightarrow 0 = j(E(\frac{1}{j}) - E(\frac{m+1}{j})) + (r - s) + 1 \Rightarrow j \mid r - s + 1$

Therefore $r - s + 1 = 0$ or $r - s + 1 = j$

If $s = 0 \Rightarrow r - s < j - 1 \Rightarrow r - s + 1 = 0 \Rightarrow E(\frac{1}{j}) = E(\frac{m+1}{j})$.

If $s = 0 \Rightarrow j \mid i - 1 \Rightarrow E(\frac{1}{j}) = E(\frac{m+1}{j} + \frac{1}{j}) = \frac{m+1}{j} = E(\frac{m+1}{j})$

With this, the theorem is already proved.

Reference:
The general term of the prime number sequence and the Smarandache prime function.

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Let's consider the function \( d(i) \) = number of divisors of the positive integer number \( i \). We have found the following expression for this function:

\[
d(i) = \sum_{k=1}^{i} E\left(\frac{i}{k}\right) - E\left(\frac{i+1}{k}\right)
\]

We proved this expression in the article "A functional recurrence to obtain the prime numbers using the Smarandache Prime Function".

We deduce that the following function:

\[
G(i) = -E\left[-\frac{d(i)-2}{i}\right]
\]

This function is called the Smarandache Prime Function (Reference). It takes the next values:

\[
G(i) = \begin{cases} 
0 & \text{if } i \text{ is prime} \\
1 & \text{if } i \text{ is compound}
\end{cases}
\]

Let is consider now \( \pi(n) \) = number of prime numbers smaller or equal than \( n \). It is simple to prove that:

\[
\pi(n) = \sum_{i=2}^{n} (1 - G(i))
\]

Let is have too:

\[
\text{If } 1 \leq k \leq p_n - 1 \Rightarrow E\left(\frac{m(k)}{n}\right) = 0 \\
\text{If } C_n \geq k \geq p_n \Rightarrow E\left(\frac{m(k)}{n}\right) = 1
\]

We will see what conditions have to carry \( C_n \).

Therefore we have the following expression for \( p_n \) n-th prime number:

\[
p_n = 1 + \sum_{k=1}^{C_n} (1 - E\left(\frac{m(k)}{n}\right))
\]

If we obtain \( C_n \) that only depends on \( n \), this expression will be the general term of the prime numbers sequence, since \( \pi \) is in function with \( G \) and \( G \) does with \( d(i) \) that is expressed in function with \( i \) too. Therefore the expression only depends on \( n \).

\( E[x] = \) The highest integer equal or less than \( n \)
Let us consider \( C_n = 2(E(n \log n) + 1) \)

Since \( p_n \sim n \log n \) from a certain \( n_0 \) it will be true that

\[
(1) \quad p_n \leq 2(E(n \log n) + 1)
\]

If \( n_0 \) it is not too big, we can prove that the inequality is true for smaller or equal values than \( n_0 \).

It is necessary to that:

\[
E\left[ \frac{n(2(E(n \log n) + 1))}{n} \right] = 1
\]

If we check the inequality:

\[
(2) \quad \pi(2(E(n \log n) + 1)) < 2n
\]

We will obtain that:

\[
\frac{n(C_n)}{n} < 2 \Rightarrow E\left[ \frac{n(C_n)}{n} \right] \leq 1 ; \quad C_n \geq p_n \Rightarrow E\left[ \frac{n(C_n)}{n} \right] = 1
\]

We can experimentally check this last inequality saying that it checks for a lot of values and the difference tends to increase, which makes to think that it is true for all \( n \).

Therefore if we prove that the next inequalities are true:

\[
(1) \quad p_n \leq 2(E(n \log n) + 1);
(2) \quad \pi(2(E(n \log n) + 1)) < 2n
\]

which seems to be very probable; we will have that the general term of the prime numbers sequence is:

\[
p_n = 1 + \sum_{k=1}^{2(E(n \log n) + 1)} \left[ 1 - E\left[ \sum_{m=1}^{k} \left( \sum_{j=1}^{\left( e^{(p_m)} - 40(p_m) log \log p_m) + 2 \right) - 1 \right) \right] \right]
\]
If now we consider the general term defined in the same way but for all real number greater than zero the following grafic is obtained:

Let us observe that if $0 < x < 1$ \( P(x) = 1 \) if $x = 1$ \( P(x) = 2 \) and for \( n - 1 < x \leq n \) \( P(x) = p_n \).

Reference:
Http://www.gallup.unm.edu/~Smarandache/primfnct.txt  
Expressions of the Smarandache Coprime Function

Sebastian Martin Ruiz, Avda de Regla, 43 Chipiona, 11550 Cadiz, Spain

Smarandache Coprime Function is defined this way:

\[ C_k(n_1, n_2, \ldots, n_k) = \begin{cases} 0 & \text{if } n_1, n_2, \ldots, n_k \text{ are coprime numbers} \\ 1 & \text{otherwise} \end{cases} \]

We see two expressions of the Smarandache Coprime Function for \( k = 2 \).

**EXPRESSION 1:**

\[ C_2(n_1, n_2) = -E\left( -\frac{n_1 n_2 - \text{lcm}(n_1, n_2)}{n_1 n_2} \right) \]

\( E(x) \) = the biggest integer number smaller or equal than \( x \).

If \( n_1, n_2 \) are coprime numbers:

\[ \text{lcm}(n_1, n_2) = n_1 n_2 \Rightarrow C_2(n_1, n_2) = -E\left( \frac{0}{n_1 n_2} \right) = 0 \]

If \( n_1, n_2 \) aren't coprime numbers:

\[ \text{lcm}(n_1, n_2) < n_1 n_2 \Rightarrow 0 < \frac{n_1 n_2 - \text{lcm}(n_1, n_2)}{n_1 n_2} < 1 \Rightarrow C_2(n_1, n_2) = 1 \]
**EXPRESSION 2:**

\[
C_2(n_1, n_2) = 1 - E \left[ \prod_{d \mid n_1} \prod_{d' \mid n_2} \frac{\mid d - d' \mid}{d > 1 \quad d' > 1} \prod_{d \mid n_1, d \mid n_2} \prod_{d (d + d')} \right]
\]

If \( n_1, n_2 \) are coprime numbers then \( d \neq d' \ \forall \ d, d' \neq 1 \)

\[
\prod_{d \mid n_1} \prod_{d' \mid n_2} \frac{\mid d - d' \mid}{d > 1 \quad d' > 1} \prod_{d \mid n_1, d \mid n_2} \prod_{d (d + d')} \Rightarrow 0 < \frac{1}{\prod_{d \mid n_1} \prod_{d \mid n_2} (d + d')} < 1 \Rightarrow C_2(n_1, n_2) = 0
\]

If \( n_1, n_2 \) aren't coprime numbers \( \exists d = d' \ d > 1, d' > 1 \Rightarrow C_2(n_1, n_2) = 1 \)

Smarandache coprime function for \( k \geq 2 \):

\[
C_k(n_1, n_2, \ldots, n_k) = -E \left[ \frac{1}{GCD(n_1, n_2, \ldots, n_k)} - 1 \right]
\]

If \( n_1, n_2, \ldots, n_k \) are coprime numbers:

\[GCD(n_1, n_2, \ldots, n_k) = 1 \Rightarrow C_k(n_1, n_2, \ldots, n_k) = 0\]

If \( n_1, n_2, \ldots, n_k \) aren't coprime numbers: \( GCD(n_1, n_2, \ldots, n_k) > 1 \)

\[
0 < \frac{1}{GCD} < 1 \Rightarrow -E \left[ \frac{1}{GCD} - 1 \right] = 1 = C_k(n_1, n_2, \ldots, n_k)
\]

**References:**

1. E. Burton, "Smarandache Prime and Coprime Functions"
The solving of the Diophantine equation

\[ 2x^2 - 3y^2 = 5 \]

i.e.,

\[ 2x^2 - 3y^2 - 5 = 0 \]

was put as an open Problem 78 by F. Smarandache in [1]. Below this problem is solved completely. Also, we consider here the Diophantine equation

\[ l^2 - 6m^2 = -5, \]

i.e.,

\[ l^2 - 6m^2 + 5 = 0 \]

and the Pellian equation

\[ u^2 - 6v^2 = 1, \]

i.e.,

\[ u^2 - 6v^2 - 1 = 0. \]

Here we use variables \( x \) and \( y \) only for equation (1) and \( l, m \) for equation (2). We will need the following denotations and definitions:

\[ \mathcal{N} = \{1, 2, 3, \ldots\}; \]

if

\[ F(t, w) = 0 \]

is an Diophantine equation, then:

(a1) we use the denotation \( < t, w > \) if and only if (or briefly: iff) \( t \) and \( w \) are integers which satisfy this equation.

(a2) we use the denotation \( < t, w > \in \mathcal{N}^2 \) iff \( t \) and \( w \) are positive integers;

\( K(t, w) \) denotes the set of all \( < t, w > \);

\( K^\circ(t, w) \) denotes the set of all \( < t, w > \in \mathcal{N}^2 \);

\( K''(t, w) = K^\circ(t, w) - \{ < 2, 1 > \}. \)

**LEMMA 1:** If \( < t, w > \in \mathcal{N}^2 \) and \( < x, y > \neq < 2, 1 > \), then there exists \( < l, m > \), such that \( < l, m > \in \mathcal{N}^2 \) and the equalities

\[ x = l + 3m \text{ and } y = l + 2m \]

hold.
LEMMA 2: Let \( <l, m> \in \mathbb{N}^2 \). If \( x \) and \( y \) are given by (1), then \( x \) and \( y \) satisfy (4) and \( <x, y> \in \mathbb{N}^2 \).

We shall note that lemmas 1 and 2 show that the map \( \varphi : K^0(l, m) \rightarrow K^0(x, y) \) given by (4) is a bijection.

Proof of Lemma 1: Let \( <x, y> \in \mathbb{N}^2 \) be chosen arbitrarily, but \( <x, y> \neq <2, 1> \). Then \( y \geq 2 \) and \( x \geq y \). Therefore,

\[
x = y + m
\]

and \( m \) is a positive integer. Subtracting (5) into (1), we obtain

\[
y^2 - 4my + 5 - 2m^2 = 0.
\]

Hence

\[
y = y_{1, 2} = 2m \pm \sqrt{6m^2 - 5}.
\]

For \( m = 1 \) (7) yields only

\[
y = y_1 = 3.
\]

indeed

\[
1 = y = y_2 < 2
\]

contradicts to \( y \geq 2 \).

Let \( m > 1 \). Then

\[
2m - \sqrt{6m^2 - 5} < 0.
\]

Therefore \( y = y_2 \) is impossible again. Thus we always have

\[
y = y_1 = 2m + \sqrt{6m^2 - 5}.
\]

Hence

\[
y - 2m = \sqrt{6m^2 - 5}.
\]

The left-hand side of (9) is a positive integer. Therefore, there exists a positive integer \( l \) such that

\[
6m^2 - 5 = l^2.
\]

Hence \( l \) and \( m \) satisfy (2) and \( <l, m> \in \mathbb{N}^2 \).

The equalities (4) hold because of (5) and (8). \( \Diamond \)

Proof of Lemma 2: Let \( <l, m> \in \mathbb{N}^2 \). Then we check the equality

\[
2(l + 3m)^2 - 3(l + 2m)^2 = 5,
\]

under the assumption of validity of (2) and the lemma is proved. \( \Diamond \)

Theorem 108 a, Theorem 109 and Theorem 110 from [2] imply the following

THEOREM 1: There exist sets \( K_i(l, m) \) such that

\[
K_i(l, m) \subset K(l, m) \quad (i = 1, 2).
\]
\( K_1(l, m) \cap K_2(l, m) = \emptyset. \)

and \( K(l, m) \) admits the representation

\[
K(l, m) = K_1(l, m) \cup K_2(l, m).
\]

The fundamental solution of \( K_1(l, m) \) is \( < -1, 1 > \) and the fundamental solution of \( K_2(l, m) \) is \( < 1, 1 >. \)

Moreover, if \( < u, v > \) runs \( K(u, v) \), then:

(a) \( < l, m > \) runs \( K_1(l, m) \) iff the equality

\[
l + m\sqrt{6} = (-1 + \sqrt{6})(u + v\sqrt{6})
\]

holds;

(b) \( < l, m > \) runs \( K_2(l, m) \) iff the equality

\[
l + m\sqrt{6} = (1 + \sqrt{6})(u + v\sqrt{6})
\]

holds.

We must note that the fundamental solution of (3) is \( < 5, 2 >. \) Let \( u_n \) and \( v_n \) be given by

\[
u_n + v_n\sqrt{6} = (5 + 2\sqrt{6})^n \quad \text{for } n \in \mathcal{N}.
\]

Then \( u_n \) and \( v_n \) satisfy (11) and \( < u_n, v_n > \in \mathcal{N}^2. \) Moreover, if \( n \) runs \( \mathcal{N} \), then \( < u_n, v_n > \) runs \( K^o(u, v) \).

Let the sets \( K_i^o(l, m) (i = 1, 2) \) are introduced by

\[
K_i^o(l, m) = K_i(l, m) \cap \mathcal{N}^2.
\]

As a corollary from the above remark and Theorem 1 we obtain

**THEOREM 2:** The set \( K^o(l, m) \) may be represented as

\[
K^o(l, m) = K_1^o(l, m) \cup K_2^o(l, m),
\]

where

\[
K_1^o(l, m) \cap K_2^o(l, m) = \emptyset.
\]

Moreover:

(c1) If \( n \) runs \( \mathcal{N} \) and the integers \( l_n \) and \( m_n \) are defined by

\[
l_n + m_n\sqrt{6} = (-1 + \sqrt{6})(5 + 2\sqrt{6})^n,
\]

then \( l_n \) and \( m_n \) satisfy (2) and \( < l_n, m_n > \) runs \( K_1^o(l, m) \);

(c2) If \( n \) runs \( \mathcal{N} \cup \{0\} \) and the integers \( l_n \) and \( m_n \) are defined by

\[
l_n + m_n\sqrt{6} = (1 + \sqrt{6})(5 + 2\sqrt{6})^n,
\]

then \( l_n \) and \( m_n \) satisfy (2) and \( < l_n, m_n > \) runs \( K_2^o(l, m) \).
Let \( \varphi \) be the above mentioned bijection. The sets \( K_i^\alpha(x, y) \) \( (i = 1, 2) \) are introduced by

\[
K_i^\alpha(x, y) = \varphi(K_i^\beta(l, m)).
\]

From Theorem 2, and especially from (14), (15), and (18) we obtain

**THEOREM 3:** The set \( K_\alpha(x, y) \) may have the representation

\[
K_\alpha(x, y) = K_1^\alpha(x, y) \cup K_2^\alpha(x, y),
\]

where

\[
K_1^\alpha(x, y) \cap K_2^\alpha(x, y) = \emptyset.
\]

Moreover:

\( d_1 \) If \( n \) runs \( N \) and the integers \( x_n \) and \( y_n \) are defined by

\[
x_n = l_n + 3m_n \quad \text{and} \quad y_n = l_n + 2m_n,
\]

where \( l_n \) and \( m_n \) are introduced by (16), then \( x_n \) and \( y_n \) satisfy (1) and \( < x_n, y_n > \) runs \( K_1^\alpha(x, y) \);

\( d_2 \) If \( n \) runs \( N \cup \{0\} \) and the integers \( x_n \) and \( y_n \) are defined again by (21), but \( l_n \) and \( m_n \) now are introduced by (17), then \( x_n \) and \( y_n \) satisfy (1) and \( < x_n, y_n > \) runs \( K_2^\alpha(x, y) \).

Theorem 3 completely solves F. Smarandache's Problem 78 from [1], because \( l_n \) and \( m_n \) could be expressed in explicit form using (16) or (17) as well.

---

Below we shall introduce a generalization of Smarandache's problem 87 from [1].

If we have to consider the Diophantine equation

\[
2x^2 - 3y^2 = p,
\]

where \( p \neq 2 \) is a prime number, then using [2, Ch. VII, exercise 2] and the same method as in the case of (1), we obtain the following result.

**THEOREM 4:**

(1) The necessary and sufficient condition for the solvability of (22) is:

\[
p \equiv 5 \pmod{24} \quad \text{or} \quad p \equiv 23 \pmod{24}
\]

(2) If (23) is valid, then there exists exactly one solution \( < x, y > \in N^2 \) of (22) such that the inequalities \( x < \sqrt{\frac{3}{2}} p; y < \sqrt{\frac{2}{3}} p \) hold. Every other solution \( < x, y > \in N^2 \) of (22) has the form:

\[
x = l + 3m \\
y = l + 2m,
\]

where \( < l, m > \in N^2 \) is a solution of the Diophantine equation

\[
l^2 - 6m^2 = -p.
\]
The question how to solve the Diophantine equation, a special case of which is the above one, is considered in Theorem 110 from [2].

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REMARK ON THE 62-th SMARANDACHE'S PROBLEM
Hristo Aladjov and Krassimir Atanassov

The 62-th problem is the following:

Let \(1 < a_1 < a_2 < \ldots\) be an infinite sequence of integers such that any three members do not constitute an arithmetic progression. Is it true that always

\[
\sum_{n \geq 1} \frac{1}{a_n} \leq 2?
\]

In [2-4] some counterexamples are given.
Easily it can be seen that the set of numbers \(\{1, 2, 4, 5, 10\}\) does not contain three numbers which are members of an arithmetic progression. On the other hand

\[
\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{10} = \frac{21}{20} > 2.
\]

Therefore, Smarandache's problem is not true in the present form, because the sum of the members of every one sequence with the above property and with first members 1, 2, 4, 5, 10 will be bigger than 2.

Some modifications of the above problem are discussed in [3,4].
We can construct the sequence which contains the minimal possible members, satisfying the Smarandache's property. The first 100 members of this sequence are:

\[
\]

In another paper the properties of this sequence will be discussed in details. Some of them are given in [3,4].
We must note that it was checked by a computer that the sum of the first 18567 members of the sequence (the 18567-th member is 4962316) is 3.000001390158..., i.e. for this sequence

\[
\sum_{n \geq 1} \frac{1}{a_n} > 3.
\]

It can be easily seen that if the first member of the sequence satisfying the Smarandache's property is not 1, or if its second member is not 2, then

\[
\sum_{n \geq 1} \frac{1}{a_n} < 3.
\]
On the other hand, there are an infinite number of sequences for which

$$\sum_{n \geq 1} \frac{1}{a_n} > 2,$$

because, for example, all sequences (their number is, obviously, infinite) generated by the above one without only one of its members will satisfy the last inequality.

This number will be discussed in the next paper of ours, too.

Now we shall cite the following unsolved problem from [2]:

*Given a sequence of integers $a_1 \leq a_2 \leq \ldots \leq a_k \leq \ldots$ where no three form an arithmetic progression, is there any bound on the sum

$$\sum_{n \geq 1} \frac{1}{a_n}?$$

From the above remark it follows that 3 is a bound of all sequences with the above property without the first sequence shown above. Some properties of this bound also will be discussed in the next paper of ours.

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The Integral Values of $\log_k S(n^k)$

Maohua Le

Abstract: Let $k$, $n$ be distinct positive integers. In this paper, we prove that $\log_k S(n^k)$ is never a positive integer.

Key words: Smarandache function, logarithm, integral value.

For any positive integer $a$, let $S(a)$ denote the Smarandache function of $a$. In [2, Problem 22], Muller posed the following problem:

Problem: Is it possible to find two distinct positive integers $k$ and $n$ such that $\log_k S(n^k)$ is a positive integer?

In this paper, we completely solve the above problem as follows:

Theorem: For any distinct positive integers $k$ and $n$, $\log_k S(n^k)$ is never a positive integer.

Proof: If $\log_k S(n^k)$ is a positive integer, then we have $k > 1$, $n > 1$ and

(1) $\log_k S(n^k) = m$,

where $m$ is a positive integer. By (1), we get

(2) $S(n^k) = k^m$.

By (1), we have

(3) $S(n^k) = S(n^{k^{\frac{1}{k}}} n) \leq S(n^{k^{\frac{1}{k}}}) + S(n) \leq \ldots kS(n)$.

Therefore, by (2) and (3), we get

(4) $k^m \leq kS(n) \leq kn$.

If $k > n > 1$, then from (4) we obtain

(5) $k^2 \leq k^m \leq kn \leq k(k-1) < k^2$

a contradiction. If $n > k > 1$, then we have

(6) $2^n \leq k^n \leq k^m \leq kn \leq (n-1)n$.

It is impossible, since $n \geq 3$. Thus, the theorem is proved.
References


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On the Functional Equation $S(n)^2 + S(n) = kn$

Rongi Chen and Maohua Le

Abstract

For any positive integer $a$, let $S(a)$ denote the Smarandache function of $a$. In this paper, we prove that the equation $S(n)^2 + S(n) = kn$ has infinitely many positive integer solutions for every positive integer $k$. Moreover, the size of the number of solutions does not depend on the parity of $k$.

Key Words: Smarandache function, functional equation, number of solutions.

1. Introduction

Let $\mathbb{N}$ be the set of positive integers. For any positive integer $a$, let

$$(1) \quad S(a) = \min \{ r \mid r \in \mathbb{N}, a \mid r! \}.$$ 

Then $S(a)$ is called the Smarandache function of $a$. Let $k$ be a fixed positive integer. In this paper we deal with the equation

$$(2) \quad S(n)^2 + S(n) = kn, \quad n \in \mathbb{N}.$$ 

For any positive integer $x$, let $N(k,x)$ denote the number of solutions $n$ with $n \leq x$, and let $N(k)$ denote the number of all solutions $n$ of (2). A computer search showed that $N(1, 10^4) = 23$, $N(2, 10^4) = 33$, $N(3, 10^4) = 20$, $N(4, 10^4) = 24$, $N(5, 10^4) = 11$ and $N(6, 10^4) = 26$. In [1] Ashbacher posed the following questions:

Question 1: Is $N(k) = \infty$ for $k = 1, 2, 3, 4, 5$ or $6$?

Question 2: Is there a positive integer $k$ for which $N(k) = 0$?

Question 3: Is there a largest positive integer for which $N(k) > 0$?

Question 4: Is there more solutions $n$ when $k$ is even than when $k$ is odd?

In this paper, we completely solve the above-mentioned questions. In fact, we prove a general result as follows:

Theorem: The positive integer $n$ is a solution of (2) if and only one of the following conditions is satisfied.

(i) $n = 1$ for $k = 2$.

(ii) $n = 4$ for $k = 5$.

(iii) $n = p(p+1)$ for $k = 1$, where $p$ is a prime with $p > 3$.

(iv) $n = p(p+1)/k$ for $k > 1$, where $p$ is a prime with $p \equiv -1 \pmod{k}$.

Corollary 1: As $x \to \infty$, we have

$$N(k,x) \sim \frac{\sqrt{x}}{(\varphi(k)\log(x))}.$$ 

Corollary 2: For any positive integers $k_1$ and $k_2$, we have

$$\frac{N(k_1)}{\varphi(k_2)} = \frac{N(k_2)}{\varphi(k_1)} \sqrt{\frac{k_1}{k_2}}.$$
By our results, we observe that (2) has infinitely many solutions $n$ for every positive integer $k$. Moreover, the size of $N(k,x)$ does not depend on the parity of $k$.

2. Preliminaries

**Lemma 1:** For any positive integers $u$ and $v$, we have $S(u) \leq S(uv)$.

**Proof:** Let $a = S(u)$ and $b = S(uv)$. By (1), $a$ and $b$ are least positive integers satisfying $u \mid a!$ and $uv \mid b!$ respectively. So we have $a \leq b$. The lemma is proved.

**Lemma 2:** For any positive integer $u$ with $u > 1$, there exists a prime factor $d$ such that $d \mid S(u)$.

**Proof:** Let $u = p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of $u$. Then, by [2], we have

$$S(u) = \max \left( S(p_1^{a_1}), S(p_2^{a_2}), \ldots, S(p_k^{a_k}) \right)$$

and $p_i \mid S(p_i^{a_i})$ for $i = 1, 2, 3, \ldots, k$. This proves the lemma.

**Lemma 3:** For any positive integer $u$, we have

$$S(u) = \begin{cases} u, & \text{if } u = 1, 4 \text{ or } p, \text{ where } p \text{ is a prime.} \\ u/2, & \text{otherwise.} \end{cases}$$

**Proof:** See [4].

**Lemma 4:** For any coprime positive integers, $u$ and $v$, we have $S(uv) = \max (S(u), S(v))$.

**Proof:** Let $a = S(u)$, $b = S(v)$ and $c = S(uv)$. By (1), $a$, $b$ and $c$ are least positive integers satisfying $u \mid a!$, $v \mid b!$ and $uv \mid c!$ respectively. This implies that $c \geq \max(a,b)$.

If $a \geq b$, then we have $u \mid a!$ and $v \mid a!$. Since $\gcd(u,v) = 1$, we get $uv \mid a!$. So we have $a \geq c$. This implies that $c = a = \max(a,b)$. By the same method, we can prove that if $a \leq b$, then $c = b = \max(a,b)$. The lemma is proved.

**Lemma 5:** For any positive number $x$, let $\Pi(x)$ denote the number of primes $p$ with $p \leq x$. As $x \to \infty$, we have $\Pi(x) \sim x/\log x$.

**Proof:** See [3].

**Lemma 6:** Let $a, b$ be integers satisfying $a > 1$ and $\gcd(a,b) = 1$. For any positive number $x$, let $\Pi(x;a,b)$ denote the number of primes $p$ such that $p \leq x$ and $p \equiv b \pmod{a}$. As $x \to \infty$, we have $\Pi(x;a,b) \sim x/\phi(a)\log x$, where $\phi(a)$ is the Euler function of $a$.

**Proof:** See [5].

3. Proofs

**Proof of Theorem:** Clearly, if $n$ satisfy (i) or (ii), then it is a solution of (2). If $n$ satisfy (iii), then $n = p(p+1)$, where $p$ is a prime with $p > 3$. Since $\gcd(p,p+1) = 1$, by Lemma 4, we get

$$S(n) = S(p(p+1)) = \max(S(p),S(p+1)).$$
Further, since \( p+1 \geq 6 \) is not a prime, by Lemma 3, we get \( S(p+1) \leq (p+1)/2 < p \). Hence, we see from (5) that \( S(n) = S(p) = p \). It implies that \( S(n)^2 + S(n) = p^2 + p = n \) and \( n \) is a solution of (2) for \( k = 1 \). By the same method, we can prove that if \( n \) satisfy the condition (iv), then it is a solution of (2) for \( k > 1 \). Thus, the sufficient condition of our theorem is proved.

We now prove the necessary condition. Let \( n \) be a solution of (2), and let \( t = S(n) \). We get from (2) that

\[
t(t+1) = kn.
\]

If \( n = 1 \) or 4, then \( t = 1 \) or 4, and \( n \) is a solution of (2) for \( k = 2 \) or 5. From below, we may assume that \( n = 1 \) or 4. Since \( \gcd(t, t+1) = 1 \), by Lemma 4, we get from (6) that

\[
S(kn) = S(t(t+1)) = \max(S(t), S(t+1)).
\]

If \( S(t) \leq S(t+1) \), then from (7) we get

\[
S(kn) = S(t+1).
\]

By Lemma 1, we have \( S(kn) \geq S(n) = t \). Hence, by (8) we obtain

\[
S(t+1) \geq t.
\]

Since \( n = 1 \) or 4, by Lemma 3, we see from (9) that either \( t = 3 \) or \( t = p-1 \), where \( p \) is a prime. When \( t = 3 \), we get \( n = 3 \) or 6. Then \( n \) satisfies the condition (iv). When \( t = p-1 \), we have \( S(n) = p-1 \) and

\[
S(kn) = S(t+1) = \max(S(t), S(t+1)).
\]

If \( S(t) > S(t+1) \), then from (17) we get

\[
S(kn) = S(t).
\]

Since \( S(kn) \geq S(n) = t \), by Lemmas 1 and 3, we see from (11) that \( S(t) = t \). Since \( n = 1 \) or 4, by Lemma 4, we get \( t = p \), where \( p \) is a prime. Hence, by (6), we obtain

\[
p(p+1) = kn.
\]

Further, since \( S(n) = p \), by Lemma 2, we have \( p \mid n \) and \( n/p \) is a positive integer. Then, by (12) we get \( p \equiv -1 \pmod{k} \). Furthermore, since \( n \neq 4 \), we get from (12) that \( p > 3 \), for \( k = 1 \). This implies that \( n \) satisfies the condition (iii) of (iv). Thus, the theorem is proved.

Proof of Corollaries 1 and 2. Let \( \Pi(x) \) and \( \Pi(x; a, b) \) be defined as in Lemmas 5 and 6 respectively. By Theorem, we have
\[
N(k,x) = \begin{cases} 
\Pi(\sqrt[4]{x + \frac{1}{4}} - \frac{1}{2}), & \text{if } k = 1, \\
\Pi(\sqrt[4]{2x + \frac{1}{4}} - \frac{1}{2}), & \text{if } k = 2, \\
\Pi(\sqrt[4]{5x + \frac{1}{4}} - \frac{1}{2}; 5,-1), & \text{if } k = 5, \\
\Pi(\sqrt[4]{kx + \frac{1}{4}} - \frac{1}{2}; k,-1), & \text{otherwise.}
\end{cases}
\]

Therefore, by Lemmas 5 and 6, we get the corollaries immediately.

References


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On The Functional Equation $Z(n) + \phi(n) = d(n)$

Zhong Li and Maohua Le

Abstract: For any positive integer $n$, let $d(n)$, $\phi(n)$ and $Z(n)$ denote the divisor function, the Euler function and the pseudo-Smarandache function of $n$ respectively. In this paper, we prove that the functional equation $Z(n) + \phi(n) = d(n)$ has no solution $n$.

Key words: divisor function, Euler function, pseudo-Smarandache function.

Let $N$ be the set of all positive integers. For any positive integer $n$, let

1. $d(n) = \sum_{d|n} 1,$

2. $\phi(n) = \sum_{1 \leq m \leq n, \gcd(m,n)=1} 1,$

3. $Z(n) = \min \left\{ a \mid a \in N, n \mid \sum_{j=1}^{a} j \right\}.$

Then $d(n)$, $\phi(n)$ and $Z(n)$ are called the divisor function, the Euler function and the Pseudo-Smarandache function of $n$ respectively. In [1], Ashbacher posed the following unsolved question:

Question: How many solutions $n$ are there to the functional equation

4. $Z(n) + \phi(n) = d(n), \ n \in N ?$

In this paper, we completely solve the above-mentioned question as follows:

Theorem: The equation $Z(n) + \phi(n) = d(n), \ n \in N$ has no solution.

Proof: Let $n$ be a solution of (4). A computer search showed that (4) has no solution with $n \leq 10000$ (see [1]). So we have $n > 10000$. Let

5. $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$

be the prime factorization of $n$. By [2, theorems 62 and 273], we see from (1), (2) and (5) that

6. $d(n) = (r_1 + 1)(r_2 + 1) \cdots (r_k + 1)$

7. $\phi(n) = n \prod_{i=1}^{k} \left( 1 - 1/p_i \right)$

On the other hand, it is a well-known fact that
(8) \( n \mid \frac{1}{2} Z(n)(Z(n) + 1) \)

(see [1]). From (8) we get

\[
Z(n) \geq \sqrt{2n + \frac{1}{4} - \frac{1}{2}}.
\]

Therefore, by (4), (5), (6), (7) and (9), we obtain

(10) \[ 1 \geq f(n) + g(n) \]

where

\[
f(n) = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)(p_i^{n/(r_i+1)}),
\]

\[
g(n) = \sqrt{2} \prod_{i=1}^{k} \left(p_i^{n/(r_i+1)} - \frac{1}{2} \prod_{i=1}^{k} 1/(r_i + 1)\right).
\]

Clearly, we see from (12) that \( g(n) > 0 \) for any positive integer \( n \) with \( n > 1 \). Hence, we get from (10) that

(13) \[ f(n) < 1. \]

If \( k = 1 \), then \( n = p_1^{11} \) and \( Z(n) \geq p_1^{11} - 1 \) by (3). Hence, by (1) and (2), \( n \) is not a solution of (4). This implies that \( k \geq 2 \).

If \( k \geq 3 \), then \( p_k \geq 5 \) and \( f(n) \geq 1 \), by (11). This contradicts with (13). So we have \( k = 2 \). Then (11) can be written as

(14) \[ f(n) = \left(1 - \frac{1}{p_1}\right)(1 - \frac{1}{p_2})\left((p_1^{r_1}p_2^{r_2})/((r_1+1)(r_2+1))\right). \]

If \( p_2 > 3 \), then from (14) we get \( f(n) \geq 1 \), a contradiction. Hence, we deduce that \( p_1 = 2 \) and \( p_2 = 3 \). Then, by (13) and (14), we obtain

(15) \[ f(n) = \left(2^{r_1}3^{r_2}\right)/3(r_1+1)(r_2+1) < 1. \]

From (15), we can calculate that \( (r_1,r_2) = (1,1) \) or \( (2,1) \). This implies that \( n \leq 12 \), a contradiction. Thus, (4) has no solution \( n \). The theorem is proved.

References


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SMARANDACHE RELATIONSHIPS AND SUBSEQUENCES*  

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ABSTRACT

Some Smarandache relationships between the terms of a given sequence are studied in the first paragraph. In the second paragraph, are studied Smarandache subsequences (whose terms have the same property as the initial sequence). In the third paragraph are studied the Smarandache magic squares and cubes of order n and some conjectures in number theory.

Key Words: Smarandache p-q relationships, Smarandache p-q-<>-subsequence, Smarandache type subsequences, Smarandache type partition, Smarandache type definitions, Smarandache type conjectures in number theory.

1) Smarandache Relationships

Let \( \{ a_n \} \) be a sequence of numbers and \( p, q \) integers \( \geq 1 \). Then we say that the terms \( a_{k+1}, a_{k+2}, \ldots, a_{k+p}, a_{k+p+1}, a_{k+p+2}, \ldots, a_{k+p+q} \) verify a Smarandache \( p-q \) relationship if

\[
a_{k+1} <> a_{k+2} <> \ldots <> a_{k+p} = a_{k+p+1} <> a_{k+p+2} <> \ldots <> a_{k+p+q}
\]

where "<>" may be any arithmetic or algebraic or analytic operation (generally a binary law on \( \{ a_1, a_2, a_3, \ldots \} \)).

If this relationship is verified for any \( k \geq 1 \) (i.e. by all terms of the sequence), then

\( \{ a_n \} \), \( n \geq 1 \) is called a Smarandache \( p-q <> \) sequence

where "<>" is replaced by "additive" if <> = +, "multiplicative" if <> = *, etc. [according to the operation (<>) used].

As a particular case, we can easily see that Fibonacci/Lucas sequence

\( (a_n + a_{n+1} = a_{n+2}) \), for \( n \geq 1 \)

is a Smarandache 2-1 additive sequence.

A Tribonacci sequence \( (a_n + a_{n+1} + a_{n+2} = a_{n+3}) \), \( n \geq 1 \) is a Smarandache 3-1 additive sequence. Etc.

Now, if we consider the sequence of Smarandache numbers,

1, 2, 3, 4, 5, 3, 7, 4, 6, 5, 11, 4, 13, 7, 5, 6, 17, \ldots ,
i.e. for each \( n \) the smallest number \( S(n) \) such that \( S(n)! \) is divisible by \( n \) [See(I)] (the values of the Smarandache Function), it raises the questions:

(a) How many quadruplets verify a Smarandache 2-2 additive relationship i.e.
\[
S(n+1) + S(n+2) = S(n+3) + S(n+4)?
\]
I found:
\[
S(6) + S(7) = S(8) + S(9), \quad 3 + 7 = 4 + 6;
S(7) + S(8) = S(9) + S(10), \quad 7 + 4 = 6 + 5;
\]
But, what about others? I am not able to tell you if there exist a finite or infinite number (?)

(b) How many quadruplets verify a Smarandache 2-2-subtractive relationship, i.e.
\[
S(n+1) - S(n+2) = S(n+3) - S(n+4)?
\]
I found:
\[
S(1) - S(2) = S(3) - S(4), \quad 1 - 2 = 3 - 4;
S(2) - S(3) = S(4) - S(5), \quad 2 - 3 = 4 - 5;
S(49) - S(50) = S(51) - S(52), \quad 14 - 10 = 17 - 13.
\]

(c) How many sextuplets verify a Smarandache 3-3 additive relationship, i.e.
\[
S(n+1) + S(n+2) + S(n+3) = S(n+4) + S(n+5) + S(n+6)?
\]
I found:
\[
S(5) + S(6) + S(7) = S(8) + S(9) + S(10), \quad 5 + 3 + 7 = 4 + 6 + 5.
\]
I read that Charles Ashbacher has a computer program that calculates the Smarandache Function's values, therefore he may be able to add more solutions to mine.

More generally:

If \( f_p \) is a \( p \)-ary relation and \( g_q \) is a \( q \)-ary relation, both of them defined on
\[
\{ a_1, a_2, a_3, \ldots \},
\]
then
\[
a_1, a_2, \ldots, a_p, a_1, a_2, \ldots, a_q
\]
verify a **Smarandache** \( f_p - g_q \) **- relationship** if
\[
f_p (a_1, a_2, \ldots, a_p) = g_q (a_1, a_2, \ldots, a_q).
\]
If this relationship is verified by all terms of the sequence, then \( \{ a_n \}, \ n \geq 1 \) is called a **Smarandache** \( f_p - g_q \)-sequence.

Study some Smarandache \( f_p - g_q \) - relationships for well-known sequences (perfect numbers, Ulam numbers, abundant numbers, Catalan numbers, Cullen numbers, etc.).

For example: a Smarandache 2-2-additive, or subtractive, or multiplicative relationship, etc.

If \( f_p \) is a \( p \)-ary relation on \( \{ a_1, a_2, a_3, \ldots \} \) and
for all \( a_k, a_k \), where \( k = 1, 2, \ldots, p \), and for all \( p \geq 1 \), then \( \{a_n\} \), \( n \geq 1 \), is called a Smarandache perfect \( f \) sequence.

If not all \( p \)-plets \((a_1, a_2, \ldots, a_p)\) and \((a_1, a_2, \ldots, a_p)\) verify the \( f \) relation, or not for all \( p \geq 1 \), the relation \( f \) is verified, then \( \{a_n\} \), \( n \geq 1 \) is called a Smarandache partial perfect \( f \)-sequence.

An example: a Smarandache partial perfect additive sequence:

\[1, 1, 0, 2, -1, 1, 1, 3, -2, 0, 2, 1, 3, 5, -4, -2, -1, 1, 1, 3, 0, 2, \ldots\]

This sequence has the property that

\[\sum_{i=1}^{p} a_i = \sum_{j=p+1}^{2p} a_j\]

for all \( p \geq 1 \).

It is constructed in the following way:

\[a_1 = a_2 = 1\]
\[a_{2p+1} = a_{p+1} - 1\]
\[a_{2p+2} = a_{p+1} + 1\]

for all \( p \geq 1 \).

(a) Can you, readers, find a general expression of \( a_n \) (as function of \( n \))?

It is periodical, or convergent or bounded?

(b) Please design (invent) yourselves other Smarandache perfect (or partial perfect) sequences.

Think about a multiplicative sequence of this type.

2) Smarandache Subsequences

Let \( \{a_n\} \), \( n \geq 1 \) be a sequence defined by a property (or a relationship involving its terms) \( P \).

Now, we screen this sequence, selecting only its terms those digits hold the property (or relationship involving the digits) \( P \).

The new sequence obtained is called:

(1) Smarandache \( P \)-digital subsequences.

For example:
(a) *Smarandache square-digital subsequence:*  
0, 1, 4, 9, 49, 100, 144, 400, 441, ...  
i.e. from 0, 1, 4, 9, 16, 25, 36, ..., \( n^2, \) ... we choose only the terms whose digits are all perfect squares  
(therefore only 0, 1, 4, and 9).  
Disregarding the square numbers of the form \( NO \ldots 0, \) where \( N \) is also a perfect square,  
how many other numbers belong to this sequence?  

(b) *Smarandache cube-digital subsequence:*  
0, 1, 8, 1000, 8000, ...  
i.e. from 0, 1, 8, 27, 64, 125, 216, ..., \( n^3, \) ... we choose only the terms whose digits are all perfect cubes  
(therefore only 0, 1, and 8).  
Similar question, disregarding the cube numbers of the form \( M0 \ldots 0 \)  
where \( M \) is a perfect cube.  

(c) *Smarandache prime digital subsequence:*  
2, 3, 5, 7, 23, 37, 53, 73, ...  
i.e. the prime numbers whose digits are all primes.  
Conjecture: this sequence is infinite.  

In the same general conditions of a given sequence, we screen it selecting only its terms whose groups of digits hold the property (or relationship involving the groups of digits) \( P. \)  
[A group of digits may contain one or more digits, but not the whole term.]  
The new sequence obtained is called:  

(2) *Smarandache P-partial digital subsequence.*  
Similar examples:  

(a) *Smarandache square-partial-digital subsequence:*  
49, 100, 144, 169, 361, 400, 441, ...  
i.e. the square members that is to be partitioned into groups of digits which are also perfect squares. (169 can be partitioned as \( 16 = 4^2 \) and \( 9 = 3^2, \) etc.)  
Disregarding the square numbers of the form \( NO \ldots 0, \) where \( N \) is also a perfect square,  
how many other numbers belong to this sequence?
(b) **Smarandache cube-partial digital subsequence:**

1000, 8000, 10648, 27000, ... 

i.e. the cube numbers that can be partitioned into groups of digits which are also perfect cubes. 

(10648 can be partitioned as $1 = 1^3$, $0 = 0^3$, $64 = 4^3$, and $8 = 2^3$).

Same question: disregarding the cube numbers of the form: 

\[ M0 \ldots 0 \]  where $M$ is also a perfect cube, how many other numbers belong to this sequence?

(c) **Smarandache prime-partial digital subsequence:**

23, 37, 53, 73, 113, 137, 173, 193, 197, ... 

i.e. prime numbers, that can be partitioned into groups of digits which are also prime, 

(113 can be partitioned as 11 and 3, both primes).

Conjecture: this sequence is infinite.

(d) **Smarandache Lucas-partial digital subsequence**

123, ... 

i.e. the sum of the two first groups of digits is equal to the last group of digits, and the whole number belongs to Lucas numbers: 

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ... 

(beginning at 2 and $L(n+2) = L(n+1) + L(n)$, $n \geq 1$) (123 is partitioned as 1, 2 and 3, then $3 = 2 + 1$).

Is 123 the only Lucas number that verifies a Smarandache type partition?

Study some Smarandache P - (partial) - digital subsequences associated to:

- Fibonacci numbers (we were not able to find any Fibonacci number verifying a Smarandache type partition, but we could not investigate large numbers; can you? Do you think none of them would belong to a Smarandache F - partial-digital subsequence? 

- Smith numbers, Eulerian numbers, Bernoulli numbers, Mock theta numbers, Smarandache type sequences etc.

**Remark:** Some sequences may not be smarandachely partitioned (i.e. their associated Smarandache type subsequences are empty).

If a sequence \( \{a_n\} \), $n \geq 1$ is defined by \( a_n = f(n) \) (a function of $n$), then a **Smarandache f-digital subsequence** is obtained by screening the sequence and selecting only its terms that can be partitioned in two groups of digits $g_1$ and $g_2$ such that $g_2 = f(g_1)$.
(3) Study similar questions for this case, which is more complex.

An interesting law may be

\[ l(a_1, a_2, \ldots, a_n) = a_1 + a_2 - a_3 + a_4 - a_5 + \ldots \]

**Smarandache prime conjecture:**

Any odd number can be expressed as the sum of two primes minus a third prime (not including the trivial solution \( p = p + q - q \) when the odd number is the prime itself).

For example:

<table>
<thead>
<tr>
<th>Odd Number</th>
<th>Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3 + 5 - 7 = 3 + 7 - 11 = 7 + 11 - 17 = 11 + 13 - 24 = ...</td>
</tr>
<tr>
<td>3</td>
<td>5 + 11 - 13 = 7 + 19 - 23 = 17 + 23 - 37 = ...</td>
</tr>
<tr>
<td>5</td>
<td>3 + 13 - 11 = ...</td>
</tr>
<tr>
<td>7</td>
<td>11 + 13 - 17 = ...</td>
</tr>
<tr>
<td>9</td>
<td>5 + 7 - 3 = ...</td>
</tr>
<tr>
<td>11</td>
<td>7 + 17 - 13 = ...</td>
</tr>
</tbody>
</table>

(a) Is this conjecture equivalent to Goldbach's conjecture (any odd number \( \geq 9 \) can be expressed as a sum of three primes - finally solved by Vinogradov in 1937 for any odd number greater than \( 3 \times 3 \))?

(b) The number of times each odd number can be expressed as a sum of two primes minus a third prime are called **Smarandache prime conjecture numbers**. None of them are known!

(c) Write a computer program to check this conjecture for as many positive odd numbers as possible.

(2) There are infinitely many numbers that cannot be expressed as the difference between a cube and a square (in absolute value).

They are called **Smarandache bad numbers(!)**

For example: 5, 6, 7, 10, 13, 14, ... are probably such bad numbers (F. Smarandache has conjectured, see[1]), while

1, 2, 3, 4, 8, 9, 11, 12, 15, ... are not, because

\[ 1 = |2^1 - 3^1| \]
\[ 2 = |3^1 - 5^1| \]
\[ 3 = |1^1 - 2^2| \]
\[ 4 = |5^1 - 11^2| \]
\[ 8 = |1^3 - 3^3| \]
\[ 9 = |6^1 - 15^2| \]
\[ 11 = |3^3 - 4^7| \]
\[ 12 = |13^3 - 4^7| \]
15 = \mid 4^3 - 7^2 \mid$, etc.

(a) Write a computer program to get as many non Smarandache bad numbers (it's easier this way!) as possible,

i.e. find an ordered array of a's such that

$$a = \mid x^3 - y^2 \mid,$$

for $x$ and $y$ integers $\geq 1$.

REFERENCES


A SET OF CONJECTURES ON SMARANDACHE SEQUENCES

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ABSTRACT

Searching through the Archives of the Arizona State University, I found interesting sequences of numbers and problems related to them. I display some of them, and the readers are welcome to contribute with solutions or ideas.

Key words: Smarandache P-digital subsequences, Smarandache P-partial subsequences, Smarandache type partition, Smarandache S-sequences, Smarandache uniform sequences, Smarandache operation sequences.

Let \( \{ a_n \} \), \( n > 1 \) be a sequence defined by a property (or a relationship involving its terms \( P \)).

Now, we screen this sequence, selecting only its terms whose digits hold the property (or relationship involving the digits) \( P \).

The new sequence obtained is called:

(1) **Smarandache P-digital subsequences**.

For example:

(a) **Smarandache square-digital subsequence**:

\[ 0, 1, 4, 9, 49, 100, 144, 400, 441, \ldots \]

i.e. from \( 0, 1, 4, 9, 16, 25, 36, \ldots, n^2, \ldots \) we choose only the terms whose digits are all perfect squares (therefore only 0, 1, 4, and 9).

Disregarding the square numbers of the form \( N0 \ldots 0 \), where \( N \) is also a perfect \( 2k \) zeros square, how many other numbers belong to this sequence?

(b) **Smarandache cube-digital subsequence**:

\[ 0, 1, 8, 1000, 8000, \ldots \]

i.e. from \( 0, 1, 8, 27, 64, 125, 216, \ldots, n^3, \ldots \) we choose only the terms whose digits are all perfect cubes (therefore only 0, 1 and 8).

Similar question, disregarding the cube numbers of the form \( M0 \ldots 0 \) \( 3k \) zeros where \( M \) is a perfect cube.

(c) **Smarandache prime digital subsequence**:

\[ 2, 3, 5, 7, 23, 37, 53, 73, \ldots \]
i.e. the prime numbers whose digits are all primes.

Conjecture: this sequence is infinite.

In the same general conditions of a given sequence, we screen it selecting only its terms whose groups of digits hold the property (or relationship involving the groups of digits) P.

[ A group of digits may contain one or more digits, but not the whole term.]

The new sequence obtained is called:

(2) Smarandache P-partial digital subsequence.

Similar examples:

(a) Smarandache square-partial-digital subsequence:

49, 100, 144, 169, 361, 400, 441, ...

i.e. the square members that is to be partitioned into groups of digits which are also perfect squares. (169 can be partitioned as $16 = 4^2$ and $9 = 3^2$, etc.)

Disregarding the square numbers of the form

\[ \text{NO \ldots 0, where N is also a perfect square,} \]

2k zeros

how many other numbers belong to this sequence?

(b) Smarandache cube-partial digital subsequence:

1000, 8000, 10648, 27000, ...

i.e. the cube numbers that can be partitioned into groups of digits which are also perfect cubes.

(10648 can be partitioned as $1 = 1^3, 0 = 0^3, 64 = 4^3$, and $8 = 2^3$).

Same question: disregarding the cube numbers of the form:

\[ \text{MO \ldots 0, where M is also a perfect cube, how many other numbers belong} \]

3k zeros

to this sequence.

(c) Smarandache prime-partial digital subsequence:

23, 37, 53, 73, 113, 137, 173, 193, 197, ...

i.e. prime numbers, that can be partitioned into groups of digits which are also prime,

(113 can be partitioned as 11 and 3, both primes).

Conjecture: this sequence is infinite.
(d) **Smarandache Lucas-partial digital subsequence**

123, ...

i.e. the sum of the two first groups of digits is equal to the last group of digits, and the whole number belongs to Lucas numbers:

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ...

(begining at 2 and \(L(n+2) = L(n+1) + L(n)\), \(n > 1\)) (123 is partitioned as 1, 2 and 3, then \(3 = 2 + 1\)). Is 123 the only Lucas number that verifies a **Smarandache type partition**?

Study some Smarandache P-(partial)-digital subsequences associated to:

- Fibonacci numbers (we were not able to find any Fibonacci number verifying a Smarandache type partition, but we could not investigate large numbers; can you? Do you think none of them would belong to a Smarandache F-partial-digital subsequence?)

- Smith numbers, Eulerian numbers, Bernoulli numbers, Mock theta numbers, Smarandache type sequences etc.

Remark: Some sequences may not be smarandachely partitioned (i.e. their associated Smarandache type subsequences are empty).

If a sequence \(\{a_n\}, n \geq 1\) is defined by \(a_n = f(n)\) (a function of \(n\)), then

**Smarandache \(f\)-digital subsequence** is obtained by screening the sequence and selecting only its terms that can be partitioned in two groups of digits \(g_1\) and \(g_2\) such that \(g_2 = f(g_1)\).

For example:

(a) If \(a_n = 2n, n \geq 1\), then

**Smarandache even-digital subsequence** is:

12, 24, 36, 48, 510, 612, 714, 816, 918, 1020, 1122, 1224, ...

(i.e. 714 can be partitioned as \(g_1 = 7, g_2 = 14\), such that \(14 = 2*7\), etc.)

(b) **Smarandache lucky-digital subsequence**

37, 49, ...

(i.e. 37 can be partitioned as 3 and 7, and \(L_3 = 7\); the lucky numbers are

1, 3, 7, 9, 13, 15, 21, 25, 31, 33, 37, 43, 49, 51, 63, ...

How many other numbers belong to this subsequence? Study the Smarandache \(f\)-digital subsequence associated to other well-known sequences.

(3) **Smarandache odd sequence**:

1, 3, 135, 1357, 13579, 1357911, 135791113, 13579111315, 1357911131517, ...

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How many of them are prime?

(4) Smarandache even sequence:

2, 24, 246, 2468, 246810, 24681012, 2468101214, 246810121416, ...

Conjecture: None of them is a perfect power!

(5) Smarandache prime sequence:

2, 23, 235, 2357, 235711, 23571113, 2357111317, 235711131719, 23571113171923, ...

How many of them are prime?

(Conjecture: a finite number).

(6) Smarandache S-sequence:

General definition:

Let \( S_1, S_2, S_3, \ldots, S_n, \ldots \) be an infinite integer sequence (noted by \( S \)). Then

\[
S_1, S_1 S_2, S_1 S_2 S_3, \ldots, S_1 S_2 S_3 \ldots S_n, \ldots
\]

is called the Smarandache S-sequence.

Question:

(a) How many of the Smarandache S-sequence belong to the initial \( S \) sequence?

(b) Or, how many of the Smarandache S-sequence verify the relation of other given sequences?

For example:

If \( S \) is the sequence of odd numbers 1, 3, 5, 7, 9, ... then the Smarandache S-sequence is 1, 13, 135, 1357, ... ([i.e. 1]) and all the other terms are odd;

Same if \( S \) is the sequence of even numbers ([i.e. 2])

The question (a) is trivial in this case.

But, when \( S \) is the sequence of primes [i.e. 3], the question becomes much harder.

Study the case when \( S \) (replaced by \( F \)) is the Fibonacci sequence (for one example):

1, 1, 2, 3, 5, 8, 13, 21, ...

Then the Smarandache F-sequence

1, 11, 112, 1123, 11235, 112358, ...

How many primes does it contain?
(7) **Smarandache uniform sequences:**

**General definition:**

Let \( n \) be an integer not equal to zero and \( d_1, d_2, \ldots, d_r \) digits in a base \( B \) (of course \( r < B \)).

Then: multiples of \( n \), written with digits \( d_1, d_2, \ldots, d_r \) only (but all \( r \) of them), in base \( B \), increasingly ordered, are called the Smarandache uniform sequence.

As a particular case it's important to study the multiples written with one digit only (when \( r = 1 \)).

**Some examples (in base 10):**

(a) Multiples of 7 written with digit 1 only:

\[ 111111, 111111, 111111, 111111, 111111, 111111, 111111, \ldots \]

(b) Multiples of 7 written with digit 2 only:

\[ 222222, 222222222222, 22222222222222222222222222, 222222222222222222222222222222222222, \ldots \]

(c) Multiples of 79365 written with digit 5 only:

\[ 555555, 555555555555, 55555555555555555555555555555555555555555555, \ldots \]

For some cases, the Smarandache uniform sequence may be empty (impossible):

(d) Multiples of 79365 written with digit 6 only (because any multiple of 79365 will end in 0 or 5).

**Remark:** If there exists at least a multiple \( m \) of \( n \), written with digits \( d_1, d_2, \ldots, d_r \) only, in base \( B \), then there exists an infinite number of multiples of \( n \) (they have the form:

\[ m, mm, mmm, mmmm, \ldots \]).

With a computer program it's easy to select all multiples (written with certain digits) of a given number - up to some limit.

**Exercise:** Find the general term expression for multiples of 7 written with digits 1, 3, 5 only in base 10.

(8) **Smarandache operation sequences:**

**General definition:**

Let \( E \) be an ordered set of elements, \( E = \{ e_1, e_2, \ldots \} \) and \( \theta \) a set of binary operations well-defined for these elements. Then:

\[ a_1 \text{ is an element of } \{ e_1, e_2, \ldots \}. \]

\[ a_{n+1} = \min \{ e_1 \theta_1 e_2 \theta_2 \ldots \theta_n e_n \} > a_n \text{, for } n > 1. \]

where all \( \theta_i \) are operations belonging to \( \theta \), is called the Smarandache operation sequence.
Some examples:

(a) When $E$ is the natural number set, and $\Theta$ is formed by the four arithmetic operations: $+$, $-$, $\times$, $\div$.

Then: $a_1 = 1$

$$a_{n+1} = \min \{ \Theta_1, \Theta_2, \ldots, \Theta_n(n+1) \} > a_n \text{, for } n > 1,$$

(therefore, all $\Theta_i$ may be chosen among addition, subtraction, multiplication or division in a convenient way).

Questions: Find this Smarandache arithmetics operation infinite sequence. Is it possible to get a general expression formula for this sequence (which starts with 1, 2, 3, 5, 4, ...)?

(b) A finite sequence

$$a_1 = 1$$

$$a_{n+1} = \min \{ \Theta_1, \Theta_2, \ldots, \Theta_{98} \} > a_n \text{, for } n > 1,$$

for $n > 1$, where all $\Theta_i$ are elements of $\{ +, -, *, / \}$.

Same questions for this Smarandache arithmetics operation finite sequence.

(c) Similarly for Smarandache algebraic operation infinite sequence

$$a_1 = 1$$

$$a_{n+1} = \min \{ \Theta_1, \Theta_2, \ldots, \Theta_n(n+1) \} > a_n \text{, for } n > 1,$$

where all $\Theta_i$ are elements of $\{ +, -, *, /, **, \sqrt[\cdot]{\cdot}, \cdot\ \cdot, \cdot, \cdot \}$

( $\times**$ means $X^Y$ and $\sqrt[\cdot]{\cdot}x$ means the $Y$th root of $x$).

The same questions become harder but more exciting.

(d) Similarly for Smarandache algebraic operation finite sequence:

$$a_1 = 1$$

$$a_{n+1} = \min \{ \Theta_1, \Theta_2, \ldots, \Theta_{98} \} > a_n \text{, for } n > 1,$$

where all $\Theta_i$ are elements of $\{ +, -, *, /, **, \sqrt[\cdot]{\cdot}, \cdot\ \cdot, \cdot, \cdot \}$

( $\times**$ means $X^Y$ and $\sqrt[\cdot]{\cdot}x$ means the $Y$th root of $x$).

Same questions.

More generally: one replaces “binary operations” by “$K_i$-ary operations” where all $K_i$ are integers $\geq 2$). Therefore,

$$a_i \in \{ e_1, e_2, \ldots \}.$$
\[ a_{n+1} = \min \{ 1, \theta_1^{(K_1)}, \theta_2^{(K_2)}, \ldots, \theta_1^{(K_1)}, K_1 \} \]

\[ \theta_1^{(K_1)} \text{ is } K_1 \text{- ary} \]

\[ \theta_2^{(K_2)} (K_2 + 1) \theta_2^{(K_2)} \ldots \theta_2^{(K_2)} (K_2 + K_2 - 1) \ldots \]

\[ \theta_2^{(K_2)} \text{ is } K_2 \text{- ary} \]

\( (n + 2 - K_1) \theta_1^{(K_1)} \ldots \theta_1^{(K_1)}(n + 1) > a_n \), for \( n \geq 1 \).

Of course \( K_1 + (K_2 - 1) + \ldots + (K_r - 1) = n+1 \).

**Remark:** The questions are much easier when \( \theta = \{ +,- \} \); study the Smarandache operation type sequences in this case.

(9) *Smarandache operation sequences at random:*

Same definitions and questions as for the previous sequences, except that

\[ a_{n+1} = \{ e_1, e_2, e_2, \ldots, e_{n+1} \} > a_n \), for \( n > 1 \),

(i.e. it's no "min" any more, therefore \( a_{n+1} \) will be chosen at random, but greater than \( a_n \), for any \( n > 1 \)).

Study these sequences with a computer program for random variables (under weak conditions).

**REFERENCES**

1. Smarandache, F. (1975) "Properties of the Numbers", University of Craiova Archives, [see also Arizona State University, Special Collections, Tempe, Arizona, USA].

ABSTRACT

Thanks to C. Dumitrescu and Dr. V. Seleacu of the University of Craiova, Department of Mathematics, I became familiar with some of the Smarandache Sequences. I list some of them, as well as questions related to them. Now I'm working in a few conjectures involving these sequences.

Examples of Smarandache Partition type sequences:

A. 1, 1, 1, 2, 2, 2, 2, 3, 4, 4, 4, ...

(How many times is $n$ written as a sum of non-null squares, disregarding the order of the terms:

for example:

\[ 9 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 \]
\[ = 1^2 + 1^2 + 1^2 + 1^2 + 2^2 \]
\[ = 1^2 + 2^2 + 2^2 \]
\[ = 3^2, \]

therefore $ns(9) = 4$.)

B. 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 4, 4, ...

(How many times is $n$ written as a sum of non-null cubes, disregarding the order of the terms:

for example:

\[ 9 = 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 \]
\[ = 1^3 + 2^3, \]

therefore, $nc(9) = 2$.)

C. General-partition type sequence:

Let $f$ be an arithmetic function and $R$ a relation among numbers.
(How many times can $n$ be written under the form:

\[ n = R(f(n_1), f(n_2), ..., f(n_k)) \]

for some $k$ and $n_1, n_2, ..., n_k$ such that

\[ n_1 + n_2 + ... + n_k = n? \]
Examples of other sequences:

(1) Smarandache Anti-symmetric sequence:

11, 1212, 123123, 12341234, 1234512345, 123456123456, 12345671234567, 1234567812345678, 123456789123456789, 1234567891012345678910, 1234567891011, 1234567891011, ...

(2) Smarandache Triangular base:

1, 2, 10, 11, 12, 100, 101, 102, 110, 1000, 1001, 1002, 1010, 1011, 10000, 10001, 10002, 10010, 10011, 10012, 100000, 100001, 100002, 100010, 100011, 100012, 100100, 1000000, 1000001, 1000002, 1000010, 1000011, 1000012, 1000100, ...

(Numbers written in the triangular base, defined as follows:

t(n) = (n(n+1))/2, for n ≥ 1.)

(3) Smarandache Double factorial base:

1, 10, 100, 101, 110, 200, 201, 1000, 1001, 1002, 1010, 1100, 1101, 1110, 1200, 10000, 10001, 10002, 10010, 10011, 10012, 10100, 10101, 10102, 10200, 10201, 11000, 11001, 11010, 11100, 11101, 11110, 11200, 11201, 12000, ...

(Numbers written in the double factorial base, defined as follows:

df(n) = n!!)

(4) Smarandache Non-multiplicative sequence:

General definition: Let $m_1$, $m_2$, ..., $m_k$ be the first $k$ terms of the sequence, where $k ≥ 2$;

then $m_i$, for $i ≥ k+1$, is the smallest number not equal to the product of $k$ previous distinct terms.

(5) Smarandache Non-arithmetic progression:

1, 2, 4, 5, 10, 11, 13, 14, 28, 29, 31, 32, 37, 38, 40, 41, 64, ...

General definition: if $m_1$, $m_2$, are the first two terms of the sequence,

then $m_k$, for $k ≥ 3$, is the smallest number such that no 3-term arithmetic progression is in the sequence.

In our case the first two terms are 1, respectively 2.

Generalization: same initial conditions, but no i-term arithmetic progression in the sequence (for a given $i ≥ 3$).
(6) Smarandache Prime product sequence:

\[ 2, 7, 31, 211, 2311, 30031, 510511, 9699691, 223092871, 6469693231, 200560490131, 7420738134811, 304250263527211, \ldots \]

\[ P_n = 1 + p_1 \cdot p_2 \cdot \ldots \cdot p_k \], where \( p_k \) is the \( k \)-th prime.

Question: How many of them are prime?

(7) Smarandache Square product sequence:

\[ 2, 5, 37, 577, 14401, 518401, 12540601, 131681894401, 159335092240001, \ldots \]

\[ S_k = 1 + s_1 \cdot s_2 \cdot \ldots \cdot s_k \], where \( s_k \) is the \( k \)-th square number.

Question: How many of them are prime?

(8) Smarandache Cubic product sequence:

\[ 2, 9, 217, 13825, 1728001, 373248001, 128024064001, 65548320768001, \ldots \]

\[ C_k = 1 + c_1 \cdot c_2 \cdot \ldots \cdot c_k \], where \( c_k \) is the \( k \)-th cubic number.

Question: How many of them are prime?

(9) Smarandache Factorial product sequence:

\[ 2, 3, 13, 289, 34561, 24883201, 125411328001, 5056584744960001, \ldots \]

\[ F_k = 1 + f_1 \cdot f_2 \cdot \ldots \cdot f_k \], where \( f_k \) is the \( k \)-th factorial number.

Question: How many of them are prime?

(10) Smarandache U-product sequence \{generalization\}:

Let \( u_n, n \geq 1 \), be a positive integer sequence. Then we define a U-sequence as follows:

\[ U_n = 1 + u_1, u_2 \cdot \ldots \cdot u_n \]

(11) Smarandache Non-geometric progression.

\[ 1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 24, 26, 27, 29, 30, 31, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 45, 47, 48, 50, 51, 53, \ldots \]

General definition: if \( m_1, m_2 \), are the first two terms of the sequence, then \( m_k \), for \( k \geq 3 \), is the smallest number such that no 3-term geometric progression is in the sequence. In our case the first two terms are 1, respectively 2.
(12) Smarandache Unary sequence:

\[ u(n) = 1 1...1, \ \text{digits of } "1", \text{ where } p_n \text{ is the } n\text{-th prime.} \]

The old question: are there are infinite number of primes belonging to the sequence?

(13) Smarandache No-prime-digit sequence:

\[ 1, 4, 6, 8, 9, 10, 11, 1, 1, 14, 1, 16, 1, 18, 19, 0, 1, 4, 6, 8, 9. \]
\[ 0, 1, 4, 6, 8, 9, 40, 41, 42, 44, 4, 46, 48, 49, 0, ... \]

(Take out all prime digits of n.)

(14) Smarandache No-square-digit-sequence.

\[ 2, 3, 5, 6, 7, 8, 2, 3, 5, 6, 7, 8, 2, 2, 22, 23, 2, 25, 26, 27, 28. \]
\[ 2, 3, 3, 32, 33, 3, 35, 36, 37, 38, 3, 2, 3, 5, 6, 7, 8, 5, 5, 52, 53, \]
\[ 5, 55, 56, 57, 58, 5, 6, 6, 62, ... \]

(Take out all square digits of n.)

SMARANDACHE CONCATENATE TYPE SEQUENCES*

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ABSTRACT

Professor Anthony Begay of Navajo Community College influenced me in writing this paper. I enjoyed the Smarandache concatenation. The sequences shown here have been extracted from the Arizona State University (Tempe) Archives. They are defined as follows:

(1) Smarandache Concatenated natural sequence:

1, 2, 3, 4, 44, 444, 4444, 44444, 444444, 4444444, 44444444, 444444444, 4444444444, 44444444444, 444444444444, 4444444444444, 44444444444444, 444444444444444, 4444444444444444, 44444444444444444, 444444444444444444, 4444444444444444444, 44444444444444444444, 444444444444444444444, 4444444444444444444444, ... 

(2) Smarandache Concatenated prime sequence:

2, 23, 235, 2357, 235711, 23571113, 2357111317, 235711131719, 23571113171923, ...

Conjecture: there are infinitely many primes among these numbers!

(3) Smarandache Concatenated odd sequence:

1, 13, 135, 1357, 13579, 1357911, 135791113, 13579111315, 1357911131517, ...

Conjecture: there are infinitely many primes among these numbers!

(4) Smarandache Concatenated even sequence:

2, 24, 246, 2468, 246810, 24681012, 2468101214, 246810121416, ...

Conjecture: none of them is a perfect power!

(5) Smarandache Concatenated S-sequence {generalization}:

Let \( s_1, s_2, s_3, s_4, \ldots, s_n, \ldots \) be an infinite integer sequence (noted by S). Then:

\[
\underbrace{s_1, s_1s_2, s_1s_2s_3, s_1s_2s_3s_4, \ldots, s_1s_2s_3s_4 \ldots s_n, \ldots}
\]

is called the Concatenated S-sequence.

Questions: (a) How many terms of the Concatenated S-sequence belong to the initial S-sequence?

(b) Or, how many terms of the Concatenated S-sequence verify the relation of other given sequences?

The first three cases are particular.

Look now at some other examples, when S is a sequence of squares, cubes, Fibonacci respectively (and one can go so on).
(6) Smarandache Concatenated Square sequence:

1, 14, 149, 14916, 1491625, 149162536, 14916253649, 1491625364964, ...

How many of them are perfect squares?

(7) Smarandache Concatenated Cubic sequence:

1, 18, 1827, 182764, 182764125, 182764125216, 182764125216343, ...

How many of them are perfect cubes?

(8) Smarandache Concatenated Fibonacci sequence:

1, 11, 112, 1123, 11235, 112358, 11235813, 1123581321, 112358132134, ...

Does any of these numbers is a Fibonacci number?

REFERENCES

   [See also Arizona State University Special Collections, Tempe, Arizona, USA].

SMARANDACHE RECURRENCE TYPE SEQUENCES∗

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ABSTRACT

Eight particular, Smarandache Recurrence Sequences and a
Smarandache General-Recurrence Sequence are defined below
and exemplified (found in State Archives, Rm, Valcea, Romania).

A. 1, 2, 5, 26, 29, 677, 680, 701, 842, 845, 866, 1517, 458330, 458333, 458354, ...

(ss2(n) is the smallest number, strictly greater than the previous one, which is the squares sum of two
previous distinct terms of the sequence; in our particular case the first two terms are 1 and 2.)

Recurrence definition:

(1) The numbers a ≤ b belong to SS2;
(2) If b, c belong to SS2, then b^2 + c^2 belongs to SS2 too;
(3) Only numbers, obtained by rules [(1) and/or (2)] applied a finite number of times, belong to SS2.

The sequence (set) SS2 is increasingly ordered.

[ Rule (1) may be changed by: the given numbers a₁, a₂, a₃, ..., aₖ, where k ≥ 2, belongs to SS2.]

B. 1, 1, 2, 4, 5, 6, 16, 17, 18, 20, 21, 22, 25, 26, 27, 29, 30, 31, 36, 37, 38, 40, 41, 42, 43, 45, 46, ...

(SS1(n) is the smallest number, strictly greater than the previous one, (for n ≥ 3), which is the squares sum
of one or more previous distinct terms of the sequence; in our particular case the first term is 1.)

Recurrence definition:

(1) The number a belongs to SS1;
(2) If b₁, b₂, ..., bₖ belong to SS1, where k ≥ 1, then b₁^2 + b₂^2 + ... + bₖ^2 belongs to SS1 too;
(3) Only numbers, obtained by rules [(1) and/or (2)] applied a finite number of times, belong to SS1.

The sequence (set) SS1 is increasingly ordered.

[ Rule (1) may be changed by: the given numbers a₁, a₂, ..., aₖ, where k ≥ 1, belong to SS1.]

C. 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, 21, ...

(NSS2(n) is the smallest number, strictly greater than the previous one, which is NOT the squares sum
of two previous distinct terms of the sequence; in our particular case the first two terms are 1 and 2.)

Recurrence definition:

(1) The numbers a ≤ b belong to NSS2;
(2) If b, c belong to NSS2, then b^2 + c^2 DOES NOT belong to NSS2; any other numbers belong to
NSS2;
(3) Only numbers, obtained by rules [(1) and/or (2)] applied a finite number of times, belong to NSS2.

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The sequence (set) NSS2 is increasingly ordered.

[Rule (1) may be changed by: the given numbers \(a_1, a_2, \ldots, a_k\), where \(k \geq 2\), belong to NSS2.]

D. 1, 2, 3, 6, 7, 8, 11, 12, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 38, 39, 42, 43, 44, 47, \ldots

\(\text{NSS1}(n)\) is the smallest number, strictly greater than the previous one, which is NOT the squares sum of one or more of the previous distinct terms of the sequence; in our particular case the first term is 1.)

Recurrence definition:

1. The number \(a\) belongs to NSS1;
2. If \(b_1, b_2, \ldots, b_k\) belong to NSS1, where \(k \geq 1\), then \(b_1^2 + b_2^2 + \ldots + b_k^2\) DOES NOT belong to NSS1; any other numbers belong to NSS1;
3. Only numbers, obtained by rules [(1) and/or (2)] applied a finite number of times, belong to NSS1.

[ Rule (1) may be changed by: the given numbers \(a_1, a_2, \ldots, a_k\), where \(k \geq 1\), belong to NSS1.]

E. 1, 2, 9, 730, 737, 389017001, 389017008, 389017729, \ldots

\(\text{CS2}(n)\) is the smallest number, strictly greater than the previous one, which is the cubes sum of two previous distinct terms of the sequence; in our particular case the first two terms are 1 and 2.)

Recurrence definition:

1. The numbers \(a \leq b\) belong to CS2;
2. If \(c, d\) belong to CS2, then \(c^3 + d^3\) belongs to CS2 too;
3. Only numbers, obtained by rules [(1) and/or (2)] applied a finite number of times, belong to CS2.

The sequence (set) CS2 is increasingly ordered.

[ Rule (1) may be changed by: the given numbers \(a_1, a_2, \ldots, a_k\), where \(k \geq 2\), belong to CS2.]

F. 1, 2, 8, 9, 10, 512, 513, 514, 520, 521, 522, 729, 730, 731, 737, 738, 739, 1241, \ldots

\(\text{CS1}(n)\) is the smallest number, strictly greater than the previous one (for \(n \geq 3\)), which is the cubes sum of one or more previous distinct terms of the sequence; in our particular case the first term is 1;

Recurrence definition:

1. The number \(a\) belongs to CS1;
2. If \(b_1, b_2, \ldots, b_k\) belong to CS1, where \(k \geq 1\), then \(b_1^3 + b_2^3 + \ldots + b_k^3\) belongs to CS2 too;
3. Only numbers, obtained by rules [(1) and/or (2)] applied a finite number of times, belong to CS1.

The sequence (set) CS1 is increasingly ordered.

[ Rule (1) may be changed by: the given numbers \(a_1, a_2, \ldots, a_k\), where \(k \geq 2\), belong to CS1.]

G. 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 36, 37, 38, \ldots

\(\text{NCS2}(n)\) is the smallest number, strictly greater than the previous one, which is NOT the cubes sum of
two previous distinct terms of the sequence; in our particular case the first two terms are 1 and 2.

Recurrence definition:

(1) The numbers \(a < b\) belong to NCS2.
(2) \([c, d] \in \text{NCS2}, \text{then } c^2 + d^2 \text{ DOES NOT belong to NCS2; any other numbers do belong to NCS2.}\)
(3) Only numbers, obtained by rules \([(1) \text{ and/or (2)}]\) applied a finite number of times, belong to NCS2.

The sequence (set) NCS2 is increasingly ordered.

[ Rule (1) may be changed by: the given numbers \(a_1, a_2, \ldots, a_k\), where \(k \geq 2\), belong to NCS2.]

H. 1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 29, 30, 31, 32, 33, 34, 37, 38, 39, ...

\(\text{NCS}_2(n)\) is the smallest number, strictly greater than the previous one, which is NOT the cubes sum of one or more previous distinct terms of the sequence; in our particular case the first term is 1.

Recurrence definition:

(1) The number \(a\) belongs to NCS1.
(2) \([b_1, b_2, \ldots, b_k] \in \text{NCS1, where } k \geq 1, \text{ then } b_1^2 + b_2^2 + \ldots + b_k^2 \text{ DOES NOT belong to NCS1.}\)
(3) Only numbers, obtained by rules \([(1) \text{ and/or (2)}]\) applied a finite number of times, belong to NCS1.

The sequence (set) NCS1 is increasingly ordered.

[ Rule (1) may be changed by: the given numbers \(a_1, a_2, \ldots, a_k\), where \(k \geq 2\), belong to NCS1.]

I. General recurrence type sequence:

General recurrence definition:

Let \(k \geq j\) be natural numbers, and \(a_1, a_2, \ldots, a_k\) be given elements, and \(R\) a \(j\)-relationship (relation among \(j\) elements).

Then:

(1) The elements \(a_1, a_2, \ldots, a_k\) belong to SGR.
(2) \([m_1, m_2, \ldots, m_j] \in \text{SGR, then } R(m_1, m_2, \ldots, m_j) \text{ belongs to SGR too.}\)
(3) Only numbers, obtained by rules \([(1) \text{ and/or (2)}]\) applied a finite number of times, belong to SGR.

The sequence (set) SGR is increasingly ordered.

Method of construction of the general recurrence sequence:

- level 1: the given elements \(a_1, a_2, \ldots, a_k\) belong to SGR;
- level 2: apply the relationship \(R\) for all combinations of \(j\) elements among \(a_1, a_2, \ldots, a_k\); the results belong to SGR too;

order all elements of levels 1 and 2 together,
level $i+1$:

if $b_1, b_2, ..., b_m$ are all elements of levels $1, 2, ..., i-1$ and $c_1, c_2, ..., c_n$ are all elements of level $i$, then apply the relationship $R$ for all combinations of $j$ elements among $b_1, b_2, ..., b_m, c_1, c_2, ..., c_n$ such that at least an element is from the level $i$;

the results belong to SGR too;

order all elements of levels $i$ and $i+1$ together;

and so on . . .

About Smarandache-Multiplicative Functions

Sabin Tabirca
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The main objective of this note is to introduce the notion of the S-multiplicative function and to give some simple properties concerning it. The name of S-multiplicative is short for Smarandache-multiplicative and reflects the main equation of the Smarandache function.

Definition 1. A function \( f: \mathbb{N}^* \rightarrow \mathbb{N}^* \) is called S-multiplicative if:

\[
(1) \quad (a,b) = 1 \Rightarrow f(ab) = \max \{ f(a), f(b) \}
\]

The following functions are obviously S-multiplicative:

1. The constant function \( f: \mathbb{N}^* \rightarrow \mathbb{N}^* \), \( f(n) = 1 \).
2. The Erdos function \( f: \mathbb{N}^* \rightarrow \mathbb{N}^* \), \( f(n) = \max \{ \pi \mid p \text{ is prime and } n \mid p \} \). [1].
3. The Smarandache function \( S: \mathbb{N}^* \rightarrow \mathbb{N} \), \( S(n) = \max \{ p \mid p^! \mid n \} \). [3].

Certainly, many properties of multiplicative functions[2] can be translated for S-multiplicative functions. The main important property of this function is presented in the following.

Definition 2. If \( f: \mathbb{N}^* \rightarrow \mathbb{N} \) is a function, then \( \overline{f}: \mathbb{N}^* \rightarrow \mathbb{N} \) is defined by

\[
\overline{f}(n) = \min \{ f(d) \mid n \mid d \}
\]

Theorem 1. If \( f \) is S-multiplicative function, then \( \overline{f} \) is S-multiplicative.

Proof. This proof is made using the following simple remark:

\[
(2) \quad (a \cdot b, (a,b)=1) \Rightarrow (\exists \ d_1 \mid a)(\exists \ d_2 \mid b)(d_1 \cdot d_2) = 1 \land d = (d_1 \cdot d_2)
\]

If \( d_1 \) and \( d_2 \) satisfy (2), then \( f(d_1 \cdot d_2) = \max \{ f(d_1), f(d_2) \} \).

Let \( a,b \) be two natural numbers, such that \( (a,b)=1 \). Therefore, we have

\[
(3) \quad \overline{f}(a \cdot b) = \min \{ f(d) \mid a \mid d \} = \min \{ f(d_1), f(d_2) \} = \min \{ \min \{ f(d_1), f(d_2) \} \mid d_1 \mid a, d_2 \mid a \}
\]

Applying the distributing property of the max and min functions, equation (3) is transformed as follows:

\[
\overline{f}(a \cdot b) = \max \{ \min \{ f(d_1), f(d_2) \} \mid d_1 \mid a, d_2 \mid a \} = \max \{ \overline{f}(a), \overline{f}(b) \}
\]

Therefore, the function \( \overline{f} \) is S-multiplicative.

We believe that many other properties can be deduced for S-multiplicative functions. Therefore, it will be in our attention to further investigate these functions.

References


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Abstract.
In this short paper I compare the Smarandache's Non-Euclidean Geometries [2] with my Orientation Table For Any Science [1].

Introduction:
Here it is An Orientation Table For Any Science (Natural or Social)

Building blocks:

S = stable (equilibrium) elements, forces, values, behavior
U = unstable (disequilibrium) elements, forces, values, behavior

Models:  

<table>
<thead>
<tr>
<th>Model</th>
<th>S</th>
<th>U</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>100%</td>
<td>0</td>
<td>A system of general stable equilibrium at its limit of perfection.</td>
</tr>
<tr>
<td>M2</td>
<td>95%</td>
<td>5%</td>
<td>A system of stable equilibrium but with minor deviations. This is the</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>methodological habitat for truths in the abstract or the pure classical</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>model in science, in the sense of Newton (physics) or Walras (economics).</td>
</tr>
<tr>
<td>M3</td>
<td>65%</td>
<td>35%</td>
<td>A mixed system of simple anomalies or relativity of the first order. The</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>equilibrium elements still prevail. Habitat for truths in the concrete.</td>
</tr>
<tr>
<td>M4</td>
<td>50%</td>
<td>50%</td>
<td>A mixed system of unstable equilibrium. In economics it represents the</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Keynesian model of &quot;equilibrium with unemployment but adding the prefix of</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>&quot;unstable&quot;. It is the usual model in modern science guided by unstable</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>equilibrium or &quot;stable disequilibrium&quot;.</td>
</tr>
<tr>
<td>M5</td>
<td>35%</td>
<td>65%</td>
<td>A mixed model of compound anomalies or relativity of the second order where</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>disequilibrium elements prevail. A weak major disequilibrium.</td>
</tr>
<tr>
<td>M6</td>
<td>5%</td>
<td>95%</td>
<td>A borderline mixed system where disequilibrium elements dominate to a</td>
</tr>
</tbody>
</table>

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My non-understandings are:

1. "Mixed Non-Euclidean Geometries" cannot mean the same thing with "Anti-Geometry". This would involve that all Non-Euclidean Geometries deny each other.

2. The "Euclidean Geometry" is just one model, specifically Model M1 on my Orientation Table. Indeed, a similar Orientation Table can be constructed for Geometry. See: enclosure: p.5

3. Independent of Model M1 (Euclidean), there is an unlimited number of possible mixed, Noneuclidean, concentrated just for study purposes in 6 other models. Only Model M7 which represents the Geometry of total disequilibrium or chaos negates model M1 and therefore may be called the Anti-Euclidean or Anti-Classical system of Geometry. Actually this is the only case when we can talk about M7 Anti-Geometry with specification.

4. The Non-Euclidean M2, M3, M4, M5 and M6 which represent a minor disequilibrium, a neutral disequilibrium, and unstable equilibrium and major disequilibria (M5, M6) systems of Geometry do not "run counter to the classical ones" (M1 with truth in the abstract and M2 with truth in the concrete) but they are just different in various degrees. There is no contradiction here or, if there is one then it is partial or imperfect but not complete.

5. To "transform the apparently unscientific ideas into scientific ones" is a treacherous operation. To me something "unscientific" means being "untrue" and I do not see how you can transform logically something which is not true into something which is true! Unless, one is willing to use a "Hocus-Pocus" logic (just a joke) or incomplete logic which closer to "Fuzzy Logic" (a more recent term). I do not know how Bertrand Russell would react to the "Fuzzy-Logic" name!

6. The term "Anti-Geometry" is not quite correct, at least not complete. Anti-Classical or -Euclidean Geometry is O.K. but with the understanding that it refers to Model M7.

References:


SMARANDACHE NUMBER RELATED TRIANGLES

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ABSTRACT

Given a triangle in Euclidean geometry it is well known that there exist an infinity of triangles each of which is similar to the given one. In Section I we make certain observations on Smarandache numbers. This enables us to impose a constraint on the lengths of the corresponding sides of similar triangles. In Section II we do this to see that infinite class of similar triangles reduces to a finite one. In Section III we disregard the similarity requirement. Finally, in Section IV we pose a set of open problems.

I SMARANDACHE NUMBERS: SOME OBSERVATIONS

Suppose a natural number n is given. The Smarandache number of n is the least number denoted by S(n) with the following property: n divides S(n)! but not (S(n)-1)!. Below is a short table containing n and S(n) for 1 ≤ n ≤ 12.

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>S(n)</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td>4</td>
<td>6</td>
<td>5</td>
<td>11</td>
<td>4</td>
</tr>
</tbody>
</table>

A look at the above table shows that S(3) = S(6) = 3, S(4) = S(8) = S(12) = 4, ....

Let a natural number k be given. Then the equation S(x) = k cannot have an infinity of solutions x. This is because the largest solution is x=k!. This observation enables us to impose a restriction on the lengths of the corresponding sides of similar triangles. In the next section we shall see how to do this. Throughout this paper the triangles are assumed to have natural number side lengths. Also, the triangles are non degenerate.

II SMARANDACHE SIMILAR TRIANGLES

Let us denote by T(a,b,c) the triangle ABC with side lengths a, b, c. Then the two similar triangles T(a,b,c) and T'(a',b',c') are said to be Smarandache Similar if S(a) = S(a'), S(b) = S(b'), S(c) = S(c'). Trivially, a given triangle is Smarandache similar to itself. Non trivially the two Pythagorean triangles (right triangles with natural number side lengths) T(3,4,5) and T'(6,8,10)
are Smarandache similar because $S(3) = S(6) = 3, S(4) = S(8) = 4, S(5) = S(10) = 5$.
However, the similar triangles $(3,4,5)$ and $(9,12,15)$ are not Smarandache similar because $S(3) = 3 * S(9)$ which is 6. In fact the class of Smarandache similar triangles generated by $T(3,4,5)$ contains just two: $T(3,4,5)$ and $T(6,8,10)$ in view of the fact that the solution set of the equation $S(x) = 3$ consists of just two members $x = 3, 6$.

For another illustration let us determine the class of Smarandache similar triangles generated by the $60^\circ$ triangle $T(a,b,c) = (5,7,8)$. The algorithm to do this is as follows: First we calculate $S(5) = 5, S(7) = 7, S(8) = 4$. Next we solve the equations $S(a') = 5, S(b') = 7, S(c') = 4$. Let us solve the last equation first.

$S(c') = 4 - c' = 4, 8, 12, 24$. Here the largest value $c' = 24 = 3c$. Hence we need not to solve the other two equations beyond the solutions $a' = 3a, b' = 3b$. This observation therefore gives us

$S(a') = 5 - a' = 5, 10, 15$ and $S(b') = 7 - b' = 7, 14, 21$.

It is now clear that the class of Smarandache similar triangles contains just two members: $(5,7,8)$ and $(15,21,24)$. 

III SMARANDACHE RELATED TRIANGLES

In this section we do not insist on the similarity requirement that we had in Section II. Hence the definition: Given a triangle $T(a,b,c)$ we say that a triangle $T'(a',b',c')$ is Smarandache related to $T$ if $S(a') = S(a), S(b') = S(b), S(c') = S(c)$. Note that the triangles $T$ and $T'$ may or may not be similar.

As an illustration let us determine all the triangles that are Smarandache related to $T(3,4,5)$. To do this we follow the same algorithm that we mentioned in Section II but we have to find all the solutions of the equations $S(a') = 3, S(b') = 4, S(c') = 5$. Therefore

$S(a') = 3 - a' = 3, 6$; $S(b') = 4 - b' = 4, 8, 12, 24$; $S(c') = 5 - c' = 5, 10, 15, 20, 30, 40, 60, 120$.

This gives us the complete solution $(a',b',c') = (3,4,5); (3,8,10); (3,12,10); (6,4,5); (6,8,5); (6,8,10); (6,12,10); (6,12,15); (6,24,20)$.

IV CONCLUSION

In the present discussion I have used small natural numbers $k$ so that the solution of the equations $S(x) = k$ can be easily determined. I do not know if this interesting converse problem of determining all natural numbers $x$ for given $k$ of the Smarandache equation $S(x) = k$ has been discussed by someone already. In case if this has not been already considered, I invite the reader to devise efficient methods to solve the preceding equation. We conclude this section by posing the following open problems to the reader.

(A) Are there two distinct dissimilar Phytagorean triangles
that are Smarandache related? i.e. both \( T(a,b,c) \) and \( T'(a',b',c') \) are Pythagorean such that \( S(a') = S(a) \), \( S(b') = S(b) \), \( S(c') = S(c) \) but \( T \) and \( T' \) are not similar.

(B) Are there two distinct and dissimilar 60° triangles (120° triangles) that are Smarandache related?

(c) Given a triangle \( T(a,b,c) \). Is it possible to give either an exact formula or an upper bound for the total number of triangles (without actually determining all of them) that are Smarandache related to \( T \)?

(D) Consider other ways of relating two triangles in the Smarandache number sense. For example, are there two triplets of natural numbers \((\alpha, \beta, \gamma)\) and \((\alpha', \beta', \gamma')\) such that \( \alpha + \beta + \gamma = \alpha' + \beta' + \gamma' = 180 \) and \( S(\alpha) = S(\alpha') \), \( S(\beta) = S(\beta') \), \( S(\gamma) = S(\gamma') \). If such distinct triplets exist the two triangles would be Smarandache related via their angles. Of course in this relationship the side lengths of the triangles may not be natural numbers.
In his recent paper[1], Sastry defines two triangles \( T(a,b,c) \) and \( T(a',b',c') \) to be Smarandache related if \( S(a) = S(a'), S(b) = S(b') \) and \( S(c) = S(c') \). The function \( S \) is known as the Smarandache function and is defined in the following way.

For \( n \) any integer greater than zero, the value of the Smarandache function \( S(n) \) is the smallest integer \( m \) such that \( n \) divides \( m! \).

He closes the paper by asking the following questions:

A) Are there two distinct dissimilar Pythagorean triangles that are Smarandache related? A triangle \( T(x,y,z) \) is Pythagorean if \( x^2 + y^2 = z^2 \).

B) Are there two distinct and dissimilar 60(120) degrees triangles that are Smarandache related? A 60(120) degrees triangle is one containing an angle of 60(120) degrees.

C) Given a triangle \( T(a,b,c) \), is it possible to give either an exact formula or an upper bound for the total number of triangles (without actually determining them), which are Smarandache related to \( T \)?

D) Consider other ways of relating two triangles in the Smarandache number sense. For example, are there two triplets of natural numbers \( (a,b,c) \) and \( (a',b',c') \) such that \( a + b + c = a' + b' + c' = 180 \) and \( S(a) = S(a'), S(b) = S(b') \) and \( S(c) = S(c') \)? If this were true, then the angles, in degrees, of the triangles would be Smarandache related.

In this paper, we will consider and answer questions (A), (B) and (D). Furthermore, we will also explore these questions using the Pseudo Smarandache function \( Z(n) \).

Given any integer \( n > 0 \), the value of the Pseudo Smarandache function is the smallest integer \( m \) such that \( n \) evenly divides

\[
\sum_{k=1}^{m} k.
\]

A) The following theorem is easy to prove.
**Theorem:** There are an infinite family of pairs of dissimilar Pythagorean triangles that are Smarandache related.

**Proof:**
Start with the two Pythagorean triangles

\[ T(3,4,5) \text{ and } T(5,12,13) \]

Clearly, these two triangles are not similar. Now, let \( p \) be an odd prime greater than 13 and form the triples

\[ T(3p,4p,5p) \text{ and } T(5p,12p,13p) \]

Obviously, these triples are also Pythagorean. It is well-known that if \( n = kp \), where \( k < p \) and \( p \) is a prime, then \( S(kp) = p \). Therefore,

\[ S(3p) = S(4p) = S(5p) = S(12p) = S(13p) = p \]

and the triples form triangles that are not similar since the originals were not. Therefore, we have the desired infinite family of solutions.

**Definition:** Given two triangles \( T(a,b,c) \) and \( T(a',b',c') \), we say that they are Pseudo Smarandache related if \( Z(a) = Z(a'), Z(b) = Z(b') \) and \( Z(c) = Z(c') \).

A computer program was written to search for dissimilar pairs of Pythagorean triples \( T(x,y,z) \) and \( T(u,v,w) \) that are also Pseudo Smarandache related. Several were found and a few are given below.

\[ x = 49, \; y = 168, \; z = 175 \]
\[ Z(x) = 48, \; Z(y) = 48, \; Z(z) = 49 \]
\[ u = 147, \; v = 196, \; w = 245 \]
\[ Z(u) = 48, \; Z(v) = 48, \; Z(w) = 49 \]

\[ x = 96, \; y = 128, \; z = 160 \]
\[ Z(x) = 63, \; Z(y) = 255, \; Z(z) = 64 \]
\[ u = 128, \; v = 504, \; w = 520 \]
\[ Z(u) = 255, \; Z(v) = 63, \; Z(w) = 64 \]

\[ x = 185, \; y = 444, \; z = 481 \]
\[ Z(x) = 74, \; Z(y) = 111, \; Z(z) = 221 \]
\[ u = 296, \; v = 555, \; w = 629 \]
\[ Z(u) = 111, \; Z(v) = 74, \; Z(w) = 221 \]

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While these numbers do not readily display the pattern of an infinite family of solutions, there is no real reason to think that there is only a finite number of solutions.

**Conjecture:** There are an infinite number of pairs of Pythagorean triples $T(x, y, z)$ and $T(u, v, w)$ that are Pseudo Smarandache related.

C) A computer program was written to search for two dissimilar 60 degrees triangles $T(a, b, c)$ and $T(a_1, b_1, c_1)$ that are Smarandache related and several solutions were found.

```
\begin{align*}
a &= 10, b = 14, c = 16, \quad S(a) &= 5, S(b) = 7, S(c) = 6 \\
a_1 &= 30, b_1 = 70, c_1 = 80, \quad S(a_1) = 5, S(b_1) = 7, S(c_1) = 6 \\

a &= 10, b = 14, c = 16, \quad S(a) = 5, S(b) = 7, S(c) = 6 \\
a_1 &= 45, b_1 = 105, c_1 = 120, \quad S(a_1) = 6, S(b_1) = 7, S(c_1) = 5 \\

a &= 16, b = 19, c = 21, \quad S(a) = 6, S(b) = 19, S(c) = 7 \\
a_1 &= 80, b_1 = 304, c_1 = 336, \quad S(a_1) = 6, S(b_1) = 19, S(c_1) = 7 \\

a &= 20, b = 28, c = 32, \quad S(a) = 5, S(b) = 7, S(c) = 8 \\
a_1 &= 60, b_1 = 140, c_1 = 160, \quad S(a_1) = 5, S(b_1) = 7, S(c_1) = 8 \\

a &= 20, b = 28, c = 32, \quad S(a) = 5, S(b) = 7, S(c) = 8 \\
a_1 &= 120, b_1 = 280, c_1 = 320, \quad S(a_1) = 5, S(b_1) = 7, S(c_1) = 8 \\
\end{align*}
```

Note that the triangles $T(30,70,80)$ and $T(45,105,120)$ are similar.

```
\begin{align*}
a &= 16, b = 19, c = 21, \quad S(a) = 6, S(b) = 19, S(c) = 7 \\
a_1 &= 80, b_1 = 304, c_1 = 336, \quad S(a_1) = 6, S(b_1) = 19, S(c_1) = 7 \\

a &= 20, b = 28, c = 32, \quad S(a) = 5, S(b) = 7, S(c) = 8 \\
a_1 &= 60, b_1 = 140, c_1 = 160, \quad S(a_1) = 5, S(b_1) = 7, S(c_1) = 8 \\

a &= 20, b = 28, c = 32, \quad S(a) = 5, S(b) = 7, S(c) = 8 \\
a_1 &= 120, b_1 = 280, c_1 = 320, \quad S(a_1) = 5, S(b_1) = 7, S(c_1) = 8 \\
\end{align*}
```

Note again that the triangles $T(60,140,160)$ and $T(120,280,320)$ are similar. Given the number of solutions found in this limited search, the following conjecture seems safe.

**Conjecture:** There are an infinite number of dissimilar 60 degrees triangles that are Smarandache related.

Another computer program was written to search for dissimilar 60 degree triangles $T(a, b, c)$ and $T(a_1, b_1, c_1)$ that are Pseudo Smarandache related. Only four pairs were found in a limited search and they are given below.

```
\begin{align*}
a &= 24, b = 56, c = 64, \quad Z(a) = 15, Z(b) = 48, Z(c) = 127 \\
a_1 &= 40, b_1 = 56, c_1 = 64, \quad Z(a_1) = 15, Z(b_1) = 48, Z(c_1) = 64 \\
\end{align*}
```
Question: Are there an infinite number of dissimilar 60 degrees triangles that are Pseudo Smarandache related?

Solutions to the corresponding problem for dissimilar 120 degrees triangles \( T(a,b,c) \) and \( T(a1,b1,c1) \) that are Smarandache related were also searched for using another computer program. Several were found, although they appear to be sparser than the corresponding 60 degrees triangles. The solutions that were found are as follows.

\[
\begin{align*}
a &= 49, \ b &= 91, \ c &= 105, \ Z(a) &= 48, \ Z(b) &= 13, \ Z(c) &= 14 \\
a1 &= 56, \ b1 &= 91, \ c1 &= 105, \ Z(a1) &= 48, \ Z(b1) &= 13, \ Z(c1) &= 14 \\
a &= 42, \ b &= 98, \ c &= 112, \ Z(a) &= 20, \ Z(b) &= 48, \ Z(c) &= 63 \\
a1 &= 70, \ b1 &= 98, \ c1 &= 112, \ Z(a1) &= 20, \ Z(b1) &= 48, \ Z(c1) &= 63 \\
a &= 42, \ b &= 98, \ c &= 112, \ Z(a) &= 20, \ Z(b) &= 48, \ Z(c) &= 63 \\
a1 &= 210, \ b1 &= 294, \ c1 &= 336, \ Z(a1) &= 20, \ Z(b1) &= 48, \ Z(c1) &= 63 \\
a &= 42, \ b &= 98, \ c &= 112, \ Z(a) &= 20, \ Z(b) &= 48, \ Z(c) &= 63 \\
a1 &= 210, \ b1 &= 294, \ c1 &= 336, \ Z(a1) &= 20, \ Z(b1) &= 48, \ Z(c1) &= 63 \\
a &= 51, \ b &= 119, \ c &= 78, \ S(a) &= 8, \ S(b) &= 14, \ S(c) &= 13 \\
a1 &= 196, \ b1 &= 364, \ c1 &= 224, \ S(a1) &= 14, \ S(b1) &= 13, \ S(c1) &= 8 \\
a &= 51, \ b &= 119, \ c &= 78, \ S(a) &= 8, \ S(b) &= 14, \ S(c) &= 13 \\
a1 &= 392, \ b1 &= 728, \ c1 &= 448, \ S(a1) &= 14, \ S(b1) &= 13, \ S(c1) &= 8 \\
a &= 51, \ b &= 119, \ c &= 78, \ S(a) &= 8, \ S(b) &= 14, \ S(c) &= 13 \\
a1 &= 119, \ b1 &= 221, \ c1 &= 136, \ S(a1) &= 17, \ S(b1) &= 17, \ S(c1) &= 17 \\
a &= 51, \ b &= 119, \ c &= 78, \ S(a) &= 8, \ S(b) &= 14, \ S(c) &= 13 \\
a1 &= 238, \ b1 &= 442, \ c1 &= 272, \ S(a1) &= 17, \ S(b1) &= 17, \ S(c1) &= 17 \\
a &= 51, \ b &= 119, \ c &= 78, \ S(a) &= 8, \ S(b) &= 14, \ S(c) &= 13 \\
a1 &= 357, \ b1 &= 663, \ c1 &= 408, \ S(a1) &= 17, \ S(b1) &= 17, \ S(c1) &= 17 \\
a &= 51, \ b &= 119, \ c &= 78, \ S(a) &= 8, \ S(b) &= 14, \ S(c) &= 13 \\
a1 &= 272, \ b1 &= 833, \ c1 &= 544, \ S(a1) &= 17, \ S(b1) &= 17, \ S(c1) &= 17 \\
a &= 51, \ b &= 119, \ c &= 78, \ S(a) &= 8, \ S(b) &= 14, \ S(c) &= 13 \\
a1 &= 476, \ b1 &= 884, \ c1 &= 544, \ S(a1) &= 17, \ S(b1) &= 17, \ S(c1) &= 17 \\
\end{align*}
\]

Note the cases where the second triangles of pairs are similar.

Question: Is there an infinite family of dissimilar 120 degrees triangles that are Smarandache related?

Finding dissimilar 120 degrees triangles that are Pseudo Smarandache related proved to be more difficult. In a search for all \( b \leq 337, a \leq 1000, c \leq 1000, a1 \leq 1000, \)
b1 ≤ 1000 and c1 ≤ 1000, only one solution,

a = 168, b = 312, c = 192, Z(a) = 48, Z(b) = 143, Z(c) = 128
a1 = 192, b1 = 588, c1 = 468, Z(a) = 128, Z(b1) = 48, Z(c1) = 143

was found.

**Question**: Is there an infinite number of dissimilar 120 degrees triangles that are Pseudo Smarandache related?

D) A computer program was written to check for triplets of natural numbers (a,b,c) and (a',b',c') such that \(a + b + c = a' + b' + c' = 180\) and \(S(a) = S(a'), S(b) = S(b')\) and \(S(c) = S(c')\) and many such pairs of triplets were found. While it is obvious that the number is finite, the following list is not exhaustive.

\[
\begin{align*}
\text{a} & = 1, \text{b} = 11, \text{c} = 168, \ S(a) = 0, S(b) = 11, S(c) = 7 \\
\text{a}' & = 1, \text{b}' = 14, \text{c}' = 165, S(a') = 0, S(b') = 7, S(c') = 11 \\
\text{a} & = 2, \text{b} = 7, \text{c} = 171, S(a) = 2, S(b) = 7, S(c) = 19 \\
\text{a}' & = 2, \text{b}' = 38, \text{c}' = 140, S(a') = 2, S(b') = 19, S(c') = 7 \\
\text{a} & = 3, \text{b} = 7, \text{c} = 170, S(a) = 3, S(b) = 7, S(c) = 17 \\
\text{a}' & = 6, \text{b}' = 21, \text{c}' = 153, S(a') = 3, S(b') = 7, S(c') = 17
\end{align*}
\]

The last being an example of a pair of triples where there is no number in common.

An exhaustive computer search revealed that all possible angle measures 1 through 178 can be an angle in such a pair of triangles except 83, 97, 107, 113, 121, 127, 137, 139, 149, 151, 163, 166, 167, 169, 172, 173, 174, 175, 176, 177, and 178.

The corresponding problem using the Pseudo Smarandache function is as follows.

Are there two triplets of natural numbers (a,b,c) and (a',b',c') such that \(a + b + c = a' + b' + c' = 180\) and \(Z(a) = Z(a'), Z(b) = Z(b')\) and \(Z(c) = Z(c')\)?

Another computer program was written that used \(Z(n)\) rather than \(S(n)\) in the search for such triples. Many solutions exist and some are given below.

\[
\begin{align*}
a & = 2, \ b = 24, \ c = 154, Z(2) = 3, Z(24) = 15, Z(154) = 55 \\
a' & = 6, \ b' = 20, \ c' = 154, Z(6) = 3, Z(20) = 15, Z(154) = 55 \\
a & = 4, \ b = 8, \ c = 168, Z(4) = 7, Z(8) = 15, Z(168) = 48 \\
a' & = 4, \ b' = 56, \ c' = 120, Z(4) = 7, Z(56) = 48, Z(120) = 15
\end{align*}
\]
The last solution shows us that there are solutions where there are no numbers common to the triples.

There are many solutions to this expression. An exhaustive computer search was performed for all possible values $1 \leq a \leq 178$ and the following numbers did not appear in any triple.


Reference

On the Difference $S(Z(n)) - Z(S(n))$
Maohua Le

Abstract: In this paper, we prove that there exist infinitely many positive integers $n$ satisfying $S(Z(n)) > Z(S(n))$ or $S(Z(n)) < Z(S(n))$.

Key words: Smarandache function, Pseudo-Smarandache function, composite function, difference.

For any positive integer $n$, let $S(n)$, $Z(n)$ denote the Smarandache function and the Pseudo-Smarandache function of $n$ respectively. In this paper, we prove the following results:

Theorem 1: There exist infinitely many $n$ satisfying $S(Z(n)) > Z(S(n))$.

Theorem 2: There exist infinitely many $n$ satisfying $S(Z(n)) < Z(S(n))$.

The above mentioned results solve Problem 21 of [1].

Proof of Theorem 1.
Let $p$ be an odd prime. If $n = (1/2)p(p+1)$, then we have

(1) $S(Z(n)) = S(Z((1/2)p(p+1))) = S(p) = p$

and

(2) $Z(S(n)) = Z(S((1/2)p(p+1))) = Z(p) = p-1$.

We see from (1) and (2) that $S(Z(n)) > Z(S(n))$ for any odd prime $p$. It is a well-known fact that there exist infinitely many odd primes $p$. Thus, the theorem is proved.

Proof of Theorem 2.
If $n = p$, where $p$ is an odd prime, then we have

(3) $S(Z(n)) = S(Z(p)) = S(p-1) < p-1$

and

(4) $Z(S(n)) = Z(S(p)) = Z(p) = p-1$.

By (3) and (4), we get $S(Z(n)) < Z(S(n))$ for any $p$. Thus, the theorem is proved.

Reference


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ABSTRACT

In this paper some definitions, examples and conjectures are exposed related to the Smarandache type functions, found in the Archives of the Arizona State University, Tempe, USA Special Collections.

(1) Smarandache Ceil Function of Second Order:

$2, 4, 3, 6, 4, 6, 10, 12, 5, 9, 14, 8, 6, 20, 22, 15, 12, 7, 10, 26, 18, 28, 30, 21, 8, 34, 12, 15, 38, 20, 9, 42, 44, 30, 46, 24, 14, 33, 10, 52, 18, 28, 58, 39, 60, 11, 62, 25, 42, 16, 66, 45, 68, 70, 12, 21, 74, 30, 76, 51, 78, 40, 18, 82, 84, 13, 57, 86, ...$

$S(n) = m$, where $m$ is the smallest positive integer for which $n$ divides $m^2$.

Reference:

(a) Surfing on the Ocean of Numbers — a few Smarandache Notions and Similar Topics, by Henry Ibstedt, Erhus University Press, Vail, USA, 1997; p. 27-30.

(2) Smarandache Ceil Function of Third Order:

$2, 2, 3, 6, 4, 6, 10, 6, 5, 3, 14, 4, 6, 10, 22, 15, 12, 7, 10, 26, 6, 14, 30, 21, 4, 34, 6, 15, 38, 20, 9, 42, 22, 30, 46, 12, 14, 33, 10, 26, 6, 28, 58, 39, 30, 11, 62, 5, 42, 8, 66, 15, 34, 70, 12, 21, 74, 30, 38, 51, 78, 20, 18, 82, 42, 13, 57, 86, ...$

$S(n) = m$, where $m$ is the smallest positive integer for which $n$ divides $m^3$.

Reference:

(a) Surfing on the Ocean of Numbers — a few Smarandache Notions and Similar Topics, by Henry Ibstedt, Erhus University Press, Vail, USA, 1997; p. 27-30.

(3) Smarandache Ceil Function of Fourth Order:

$2, 2, 3, 6, 2, 6, 10, 6, 5, 3, 14, 4, 6, 10, 22, 15, 6, 7, 10, 26, 6, 14, 30, 21, 4, 34, 6, 15, 38, 10, 3, 42, 22, 30, 46, 12, 14, 33, 10, 26, 6, 14, 58, 39, 30, 11, 62, 5, 42, 4, 66, 15, 34, 70, 6, 21, 74, 30, 38, 51, 78, 20, 6, 82, 42, 13, 57, 86, ...$

$S(n) = m$, where $m$ is the smallest positive integer for which $n$ divides $m^4$.

Reference:

(a) Surfing on the Ocean of Numbers — a few Smarandache Notions and Similar Topics, by Henry Ibstedt, Erhus University Press, Vail, USA, 1997; p. 27-30.
(4) Smarandache Ceiling Function of Fifth Order:

\[ S(n) = m, \text{ where } m \text{ is the smallest positive integer for which } n \text{ divides } m^5. \]

Reference:
(a) Surfing on the Ocean of Numbers -- a few Smarandache Notions and Similar Topics, by Henry Ibstedt, Erhus University Press, Vail, USA, 1997; p. 27-30.

(5) Smarandache Ceiling Function of Sixth Order:

\[ S(n) = m, \text{ where } m \text{ is the smallest positive integer for which } n \text{ divides } m^6. \]

Reference:
(a) Surfing on the Ocean of Numbers -- a few Smarandache Notions and Similar Topics, by Henry Ibstedt, Erhus University Press, Vail, USA, 1997; p. 27-30.

(6) Smarandache - Fibonacci triplets:

\[ 11, 121, 4902, 26245, 32112, 64010, 368140, 415664, 2091206, 2519648, 4573053, 7783364, 79269727, 136193976, 321022289, 445810543, 559199345, 670994143, 836250239, 893950202, 937203749, 1041478032, 1148788154, \ldots \]

(An integer \( n \) such that \( S(n) = S(n-1) + S(n-2) \) where \( S(k) \) is the Smarandache function: the smallest number \( k \) such that \( S(k)! \) is divisible by \( k \).)

Remarks:
It is not known if this sequence has infinitely or finitely many terms.
H. Ibstedt and C. Ashbacher independently conjectured that there are infinitely many.
H. I. found the biggest known number: 19 448 047 080 036.

References:
(a) Surfing on the Ocean of Numbers -- a few Smarandache Notions and Similar Topics, by Henry Ibstedt, Erhus University Press, Vail, USA, 1997; p. 19-23.
(7) Smarandache-Radu duplets

224, 2057, 265225, 843637, 6530355, 24652435, 35558770, 40201975, 45388758,
46297822, 67697937, 138852445, 157906534, 171531580, 299441785, 551787925,
1223918824, 1276553470, 1655870629, 1853717287, 1994004499, 2256222280, ...

(An integer n such that between S(n) and S(n+1) there is no prime [S(n) and
S(n + 1) included].

where S(k) is the Smarandache function: the smallest number k such that S(k)!
is divisible by k.)

Remarks:

It is not known if this sequence has infinitely or finitely many terms.

H. Ibstedt conjectured that there are infinitely many.

H. I. found the biggest known number:

270 329 975 921 205 253 634 707 051 822 848 570 391 313!

References:

(a) Surfing on the Ocean of Numbers -- a few Smarandache Notions and Similar
Topics, by Henry Ibstedt, Erhus University Press, Vail, USA, 1997; p. 19-23.

(b) I. M. Radu, <Mathematical Spectrum>, Sheffield University, UK, Vol. 27, (No.
2), 1994/5; p. 43.

The Smarandache Deconstructive Sequence (SDS(n)) of integers is constructed by sequentially repeating the digits 1-9 in the following way:

1, 23, 456, 7891, 23456, 789123, 4567891, 23456789, 123456789, 1234567891, ...

and first appeared in the collection by Smarandache[1]. In a later collection by Kashihara[2], the question was asked:

How many primes are there in this sequence?

In this article, we will briefly explore that question and raise a few others concerning this sequence.

The values of the first thirty elements of this sequence appear in Table 1. From the list, it seems clear that the trailing digits repeat the pattern,

1, 3, 6, 1, 6, 3, 1, 9, 9, 1, 3, 6, 1, 6, 3, 1, 9, 9, 1, ...

and it is simple to prove that this is indeed the case. Given the rules used in the construction of this sequence, the remaining columns also have similar patterns.

It is also simple to prove that every third element in the sequence is evenly divisible by 3. More specifically, 3 | SDS(n) if and only if 3 | n.

The list contains the eight primes

23, 4567891, 23456789, 123456789, 4567891234567891, 1234567891234567891234567891.

If we do not consider the first element in the list, the percentage of primes is $\frac{8}{29} = 0.276$.

Given this, admittedly slim, numeric evidence and the regular nature of the digits, the author is confident enough to offer the following conjecture.

**Conjecture 1:** The Smarandache Deconstructive Sequence contains an infinite number of primes.

Two out of every nine numbers end in 6. In examining the factorizations of these numbers, we see that 456 is divisible by $2^4$, 23456 by $2^5$, and all others by $2^7$. This prompts the question:
Table 1.

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**Question 1:** Does every even element of the Smarandache Deconstructive Sequence contain at least three instances of the prime 2 as a factor?

Even more specifically,

**Question 2:** If we form a sequence from the elements of SDS(n) that end in a 6, do the powers of 2 that divide them form a monotonically increasing sequence?

The following is prompted by examining the divisors of the elements of the sequence.
Question 3: Let $k$ be the largest integer such that $3^k \mid n$ and $j$ the largest integer such that $3^j \mid SDS(n)$. Is it true that $k$ is always equal to $j$?

And we close with the question

Question 4: Are there any other patterns of divisibility in this sequence?

* This paper originally appeared in *Journal of Recreational Mathematics*, Vol. 29, No. 2.

References

S-PRIMALITY DEGREE OF A NUMBER AND S-PRIME NUMBERS

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Abstract.
In this paper we define the S-Primality Degree of a Number, the S-Prime Numbers, and make some considerations on them.

The depths involved by the Smarandache function are far from being exhausted or completely explored. If one takes \( S(1) = 1 \) then

\[
\sum_{1 \leq n \leq x} \left\lfloor \frac{S(n)}{n} \right\rfloor = \begin{cases} 
\pi(x) + 1, & \text{if } 1 \leq x < 4; \\
\pi(x) + 2, & \text{if } x \geq 4;
\end{cases}
\]

where \( S(n) \) is the Smarandache function, \( \pi(x) \) the number of primes less than or equal to \( x \), and \( \left\lfloor a \right\rfloor \) the greatest integer less than or equal to \( a \) (integer part).

The ratio \( \frac{S(n)}{n} \) measures the S-Primality Degree (S stands for Smarandache) of the number \( n \).

Whereas \( n \) is called S-Prime if \( \frac{S(n)}{n} = 1 \).

Therefore, the set of S-Prime numbers is \( P \cup \{1, 4\} \), with \( P = \{2, 3, 5, 7, 11, 13, 17, \ldots\} \) the set of traditional prime numbers.

Traversing the natural number set \( N' = \{1, 2, 3, 4, 5, 6, \ldots\} \) we meet "the most composite" numbers (= the most "broken up"), i.e. those of the form \( n = k! \) for which \( \frac{S(k!)}{k!} = k/k! = 1/(k-1)! \).

The philosophy of this classification of the natural numbers is that number 4, for example, appears as a prime (S-Prime) and in the same time composite (broken up).

It is not surprising that in the approachment of Fermat Last Theorem's proof, \( x^y + y^z = z^w \) doesn't have nonzero integer solutions for \( n \geq 3 \), it had had to treat besides the cases \( n \in \{3, 5, 7, 11, 13, 17, \ldots\} \) the special case \( n = 4 \) as well because, for example, \( x^2 + y^2 = z^2 \)

is reducible to \( (x^2) + (y^2) = (z^2) \).

Also, it is not surprising that Einstein (intuitively) choose the \( R^4 \) space to treat the relativity theory.

It is not surprising that the multiplication of triplets \( (a,b,c)(m,n,p) \) does not really work when we want to sink \( R^2 \) into \( R^3 \), while the multiplication of quadruplets \( (a,b,c,d)(m,n,p,q) \) led to the quaternions theory.

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Some Elementary Algebraic Considerations Inspired by Smarandache Type Functions (II)

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Abstract

The paper presents new properties for some functions constructed similarly to the function \( \eta : \mathbb{N}^* \rightarrow \mathbb{N}^* \), the Smarandache function, defined by \( \forall n \in \mathbb{N}^*, \eta(n) = \min \{ k | k! \text{ is divisible by } n \} \) — "Smarandache's type function".

The Smarandache \( \eta \) function and its principal properties are already known in the literature of speciality. Other functions were built analogously, among which the following ones.

The function \( \eta_1 \). Starting from a sequence of positive integers \( \sigma : \mathbb{N}^* \rightarrow \mathbb{N}^* \) satisfying the condition

\[ \forall n \in \mathbb{N}^*, \exists m_n \in \mathbb{N}^*, \forall m \geq m_n \Rightarrow n / \sigma(m) \] (1)

an associated function was built \( \eta_1 : \mathbb{N}^* \rightarrow \mathbb{N}^* \), defined by

\[ \eta_1(n) = \min \{ m_n | m_n \text{ is given by (1)} \} , \forall n \in \mathbb{N}^* . \] (2)

Such sequences - possibly satisfying an extra condition - considered by G. Christol to generalise the p-adic numbers were called also multiplicative convergent to zero (m.c.z.). An example is \( \sigma : \mathbb{N}^* \rightarrow \mathbb{N}^* \) with \( \sigma(n) = n! \). For \( n = 6 \), there is \( m_6 = 4 \) such that \( \forall m \geq 4 \Rightarrow 6 / m! \) (6/4! for \( m = 4 \); 6/5! for \( m = 5 \)) but there is and \( m_6 = 7 \) such that \( \forall m \geq 7 \Rightarrow 6 / m! \); because the smallest of them is \( m_6 = 3 \) such that \( \forall m \geq 3 \Rightarrow 6 / 3! \), it results \( \eta_1(6) = 3 \).
We note that for \( \sigma(n) = n! \) the associated function \( \eta_1 \) is just the \( \eta \) function - from where the idea of building the \( \eta_i \) functions (by generalization of the sequence).

The function \( \eta_2 \). A sequence of positive integers \( \sigma: \mathbb{N}^* \rightarrow \mathbb{N}^* \) is called "of divisibility sequence (d.s.)" if:

\[
m/n = \sigma(m)/\sigma(n),
\]

and "of strong divisibility sequence (s.d.s.)" if

\[
\sigma((m, n)) = (\sigma(m), \sigma(n)), \quad \forall m, n \in \mathbb{N}^*,
\]

\((m, n)\) being the greatest common factor.

(Strong divisibility sequences are studied for instance by N. Jensen in [5]. It is known that the Fibonacci sequence is a s.d.s.).

Starting from a sequence \( \sigma: \mathbb{N}^* \rightarrow \mathbb{N}^* \) satisfying the condition

\[
\forall n \in \mathbb{N}^*, \exists m_n \in \mathbb{N}^*, \forall m \in \mathbb{N}^*, m_n/m \Rightarrow n/\sigma(m)
\]

an associated function was built that is \( \eta_2: \mathbb{N}^* \rightarrow \mathbb{N}^* \) defined by

\[
\eta_2(n) = \min \{ m_n | m_n \text{ is given by (5)} \}, \quad \forall n \in \mathbb{N}^*.
\]

If the sequence \( \sigma \) is d.s. or s.d.s., the function \( \eta_2 \) has new properties with interesting algebraic interpretations.

We observe that in (1) appeared both the natural order \( m \geq m_n \) and the divisibility as relation of order on \( \mathbb{N}^* \) \( (n/\sigma(m)) \) and in (5), only the divisibility as relation of order on \( \mathbb{N}^* \). From the alternation of the two relations of order on \( \mathbb{N}^* \) can be defined analogously two more functions \( \eta_3 \) and \( \eta_4 \). (see [1])

Starting from a sequence \( \sigma: \mathbb{N}^* \rightarrow \mathbb{N}^* \) satisfying the condition

\[
\forall n \in \mathbb{N}^*, \exists m_n \in \mathbb{N}^*, \forall m \in \mathbb{N}^*, m_n/m \Rightarrow n \leq \sigma(m)
\]

an associated function was built that is \( \eta_3: \mathbb{N}^* \rightarrow \mathbb{N}^* \), defined by

\[
\eta_3(n) = \min \{ m_n | m_n \text{ is given by (7)} \}, \quad \forall n \in \mathbb{N}^*.
\]

Also, starting from a sequence \( \sigma: \mathbb{N}^* \rightarrow \mathbb{N}^* \) satisfying the condition

\[
\forall n \in \mathbb{N}^*, \exists m_n \in \mathbb{N}^*, \forall m \in \mathbb{N}^*, m_n \leq m \Rightarrow n \leq \sigma(m)
\]

an associated function was built that is \( \eta_4: \mathbb{N}^* \rightarrow \mathbb{N}^* \), defined by

\[
\eta_4(n) = \min \{ m_n | m_n \text{ is given by (9)} \}, \quad \forall n \in \mathbb{N}^*.
\]

The principal properties of the functions above are divided in three groups:
I The arithmetical properties of the proper function.

II The properties of sumatory function associated to each of the numerical functions above. (see [3])

III The algebraical properties of the proper function. Thanks to the arithmetical properties, every function can be viewed as morphism (endomorphism) between certain universal algebras (we can be obtain several situations considering various operations of \( N^* \)). (see [2], [4])

This paper presents a construction from group III which guides to a prolongation \( s_4 \) of the function \( \eta_4 \) for more complexe universal algebras.

If the initial sequence is s.d.s., the associated function \( \eta_4 \) has a series of important properties from which we retain:

\[
\begin{align*}
\eta_4(\max\{a, b\}) &= \max\{\eta_4(a), \eta_4(b)\}; \\
\eta_4(\min\{a, b\}) &= \min\{\eta_4(a), \eta_4(b)\} \forall a, b \in N^*. 
\end{align*}
\]

We may stand out, from other possible structures on \( N^* \), the universal algebra \((N^*, \Omega)\) where the set of operations is \( \Omega = \{\vee, \wedge, \psi_0\} \) with \( \vee, \wedge : (N^*)^2 \to N^* \) defined by \( a \vee b = \sup\{a, b\} \), \( a \wedge b = \inf\{a, b\} \), \( \forall a, b \in N^* \) (\( N^* \) is a lattice with the natural order) and \( \psi_0 : (N^*)^0 \to N^* \) - a null operation that fixes 1, the unique particular element with the role of neutral element for "\( \vee \)"; \( 1 = e_\vee \).

Therefore, the universal algebra \((N^*, \Omega)\) is of type \( \tau = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{pmatrix} = (2, 2, 0) \).

With the properties (11) and (12) the function \( \eta_4 \) is endomorphism for the universal algebra above. It can be stated

**Teorema 1** If \( \eta_4 : N^* \to N^* \) is the function defined by (10), endomorphism for the universal algebra \((N^*, \Omega)\) and \( I \) is a set, then there is a unique \( s_4 : (N^*)^I \to (N^*)^I \), endomorphism for the universal algebra \(((N^*)^I, \Omega)\) so that \( p_i \circ s_4 = \eta_4 \circ p_i, \forall i \in I \), where \( p_j : (N^*)^I \to N^* \) with \( \forall a \in \{a_i\}_{i \in I} \in (N^*)^I \), \( p_j(a) = a_j, \forall j \in I \), are the canonical projections, morphisms between \(((N^*)^I, \Omega)\) and \((N^*, \Omega)\).

The proof can be done directly: it is shown that the correspondence \( \eta_4 \) is a function. endomorphism and complies with the required
conditions. The operations of $\Omega$ for the universal algebra $\left( (N^*)^I, \Omega \right)$ are made "on components".

The algebraic properties of $s_4$ - the prolongation to more ampler universal algebra of the function $\eta_4$ - for its restriction to $N^*$, could bring new properties for the function $\eta_4$ that we considered above.

The paper contents, in completion, a formula of calcul for the sumatory function $F_{\eta_2}$ of function $\eta_2$.

If the initial sequence is s.d.s., this formula is:

$$
F_{\eta_2}(n) = \eta_2(1) + \sum_{h, t=1 \atop h \neq t}^k [\eta_2(p_h), \eta_2(p_t)] + \sum_{h, t, q=1 \atop h \neq t \neq q}^k [\eta_2(p_h), \eta_2(p_t), \eta_2(p_q)] + \cdots + \eta_2(n), \ \forall n = p_1 \cdot p_2 \cdots p_k, \ p_i, \ i = 1, k - \text{prime numbers}
$$

and $\eta_2(p^a) = F_{\eta_2}(p^a) - F_{\eta_2}(p^{a-1})$.

References


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ON THE FUNCTIONAL EQUATION

\[(S(n))^r + (S(n))^{r-1} + \cdots + S(n) = n\]

Rongji Chen

Abstract For any positive integer \(n\), let \(S(n)\) be the Smarandache function of \(n\). Let \(r\) be a fixed positive integer with \(r \geq 3\). In this paper we give a necessary and sufficient condition for the functional equation \((S(n))^r + (S(n))^{r-1} + \cdots + S(n) = n\) to have positive integer solutions \(n\).

Key words Smarandache function, functional equation, solvability.

1 Introduction

Let \(\mathbb{N}\) be the set of all positive integers. For any \(n \in \mathbb{N}\), let the arithmetic function
\[
S(n) = \min\{a \mid a \in \mathbb{N}, n \mid a!\}
\]
Then \(S(n)\) is called the Smarandache function of \(n\). For a fixed \(r \in \mathbb{N}\) with \(r \geq 3\), we discuss the solvability of the functional equation
\[
(S(n))^r + (S(n))^{r-1} + \cdots + S(n) = n, n \in \mathbb{N}
\]
There are many unsolved questions concerned this equation (see [1]). A computer search showed that if \(r = 3\), then (2) has no solution \(n\) with \(n \leq 10000\). In this paper we prove a general result as follows.
Theorem  For any fixed \( r \in \mathbb{N} \) with \( r \geq 3 \), a positive integer \( n \) is a solution of (2) if and only if \( n = p^{r-1} + p^{r-2} + \cdots + 1 \), where \( p \) is an odd prime satisfying \( p^{r-1} + p^{r-2} + \cdots + 1 \mid (p - 1)! \).

By our theorem, we find that if \( r = 3 \), then (2) has exactly two solutions \( n = 305319 \) and \( n = 499359 \) with \( n < 1000000 \).

2 Preliminaries

Lemma 1  For any \( u, v \in \mathbb{N} \) with \( \gcd(u, v) = 1 \), we have \( S(uv) = \max(S(u), S(v)) \).

Proof  Let \( a = S(u) \), \( b = S(v) \) and \( c = S(uv) \). By (1), \( a, b, c \)
are least positive integers satisfying
\[
3 \quad u \mid a!, \quad v \mid b!, \quad uv \mid c!,
\]
respectively. We see from (3) that
\[
4 \quad c \geq \max(a, b)
\]
If \( a \geq b \), then \( u \mid a! \) and \( v \mid a! \) by (3). Since \( \gcd(u, v) = 1 \), we get \( uv \mid a! \). It implies that \( a \geq c \). Therefore, by (4), we obtain \( c = a = \max(a, b) \). By the same method, we can prove that if \( a \leq b \), then \( c = b = \max(a, b) \). The lemma is proved.

Lemma 2  If \( S(u) = u \), then \( u = 1, 4 \) or \( p \), where \( p \) is a prime.

Proof  See [3].

Lemma 3  If \( u > 1 \), where \( u \in \mathbb{N} \), then \( u \) has a prime factor \( p \) such that \( p \mid S(u) \).

Proof  Let \( u = p_1^{k_1} p_2^{k_2} \cdots p_k^{k_k} \) be the factorization of \( u \). It is a well known fact that \( S(u) = \max(S(p_1^{k_1}), S(p_2^{k_2}), \ldots, S(p_k^{k_k})) \) and \( p_i \mid S(p_i^{k_i}) \) for \( i = 1, 2, \ldots, k \) (see [2]). The lemma follows immediately.
3 Proof of Theorem

Let \( n = p(p^{r-1} + p^{r-2} + \cdots + 1) \), where \( p \) is an odd prime satisfying \( p^{r-1} + p^{r-2} + \cdots + 1 \mid (p - 1)! \). Then, by (1), we get \( S(n) = p \). Therefore, \( n \) is a solution of (2).

On the other hand, let \( n \) be a solution of (2). Then we have \( n > 1 \). Further, let \( t = S(n) \). We get from (2) that

\[
(5) \quad t(t^{r-1} + t^{r-2} + \cdots + 1) = n.
\]

Since \( \gcd(t, t^{r-1} + t^{r-2} + \cdots + 1) = 1 \), by Lemma 1, we see from (5) that

\[
(6) \quad t = S(n) = S(t(t^{r-1} + t^{r-2} + \cdots + 1))
\]

\[
= \max(S(t), S(t^{r-1} + t^{r-2} + \cdots + 1))
\]

If \( S(t) \leq S(t^{r-1} + t^{r-2} + \cdots + 1) \), then from (6) we get

\[
(7) \quad t = S(t^{r-1} + t^{r-2} + \cdots + 1)
\]

Since \( t^{r-1} + t^{r-2} + \cdots + 1 > 1 \), by Lemma 3, \( t^{r-1} + t^{r-2} + \cdots + 1 \) has a prime factor \( p \) such that \( p \mid S(t^{r-1} + t^{r-2} + \cdots + 1) \). Hence, by (7), we get \( p \mid t \). However, since \( \gcd(t, t^{r-1} + t^{r-2} + \cdots + 1) = 1 \), it is impossible. So we have

\[
(8) \quad S(t) > S(t^{r-1} + t^{r-2} + \cdots + 1)
\]

and

\[
(9) \quad t = S(t),
\]

by (6).

On applying Lemma 2, we see from (9) that either \( t = 4 \) or \( t = p \), where \( p \) is a prime. If \( t = 4 \), then \( n = 4, 8, 12 \) or 24. However, since \( r > 3 \), we get from (5) that \( t^r + t^{r-1} + \cdots + t \geq 4^3 + 4^2 + 4 > 24 > n \), a contradiction. If \( t = p \), then from (8) and (9) we obtain

\[
(10) \quad S(p^{r-1} + p^{r-2} + \cdots + 1) < S(t) = S(p) = p
\]
It implies that $p^{r-1} + p^{r-2} + \cdots + 1 \mid (p - 1)!$ and $p > 2$. Therefore, we see from (5) that if $n$ is a solution of (2), then $n = p(p^{r-1} + p^{r-2} + \cdots + 1)$, where $p$ is an odd prime satisfying $p^{r-1} + p^{r-2} + \cdots + 1 \mid (p - 1)!$. Thus, the theorem is proved.

References


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SOLUTION OF TWO QUESTIONS CONCERNING
THE DIVISOR FUNCTION AND THE PSEUDO-
SMARANDACHE FUNCTION
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Abstract In this paper we completely solve two questions concerning the divisor function and the pseudo-Smarandache function.

Key words divisor function, pseudo-Smarandache function, functional equation

1 Introduction

Let $\mathbb{N}$ be the set of all positive integers. For any $n \in \mathbb{N}$, let

$$d(n) = \sum_{d \mid n} 1,$$

$$Z(n) = \min \{a \mid a \in \mathbb{N}, n \mid \sum_{j=1}^{a} j\}$$

Then $d(n)$ and $Z(n)$ are called the divisor function and the pseudo-Smarandache function of $n$, respectively. In [1], Ashbacher posed the following unsolved questions.

Question 1 How many solutions $n$ are there to the functional equation.

$$Z(n) = d(n), n \in \mathbb{N}?$$

Question 2 How many solutions $n$ are there to the functional equation.
In this paper we completely solve the above questions as follows.

**Theorem 1**  The equation (3) has only the solutions $n = 1, 3$ and $10$.

**Theorem 2**  The equation (4) has only the solution $n = 56$.

### 2 Proof of Theorem 1

A computer search showed that (3) has only the solutions $n = 1, 3$ and $10$ with $n \leq 10000$ (see [1]).

We now let $n$ be a solution of (3) with $n \neq 1, 3$ or 10. Then we have $n > 10000$. Let

$$n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$$

be the factorization of $n$. By [2, Theorem 273], we get from (1) and (5) that

$$d(n) = (r_1 + 1)(r_2 + 1) \cdots (r_k + 1).$$

On the other hand, since $\sum_{j=1}^{a} j = a(a + 1)/2$ for any $a \in \mathbb{N}$, we see from (2) that $n | Z(n)(Z(n) + 1)/2$. It implies that $Z(n)(Z(n) + 1)/2 \geq n$. So we have

$$Z(n) \geq \sqrt{2n + \frac{1}{4} - \frac{1}{2}}$$

Hence, by (3), (5), (6) and (7), we get

$$1 > \sqrt{2} \prod_{i=1}^{k} \frac{p_i^{r_i/2}}{r_i + 1} - \frac{1}{2} \prod_{i=1}^{k} \frac{1}{r_i + 1}$$

If $p_1 > 3$, then from (8) we get $p_1 \geq 5$ and

$$1 \geq \sqrt{2} \left(\frac{\sqrt{5}}{2}\right)^k - \frac{1}{2^{k+1}} > 1,$$
a contradiction. Therefore, if (8) holds, then either \( p_1 = 2 \) or \( p_1 = 3 \). By the same method, then \( n \) must satisfy one of the following conditions.

(i) \( p_1 = 2 \) and \( r_1 \leq 4 \).
(ii) \( p_1 = 3 \) and \( r_1 = 1 \).

However, by (8), we can calculate that \( n < 10000 \), a contradiction. Thus, the theorem is proved.

3 Proof of Theorem 2

A computer search showed that (4) has only the solution \( n = 56 \) with \( n \leq 10000 \) (see \([1]\)). We now let \( n \) be a solution of (4) with \( n \neq 56 \). Then we have \( n > 10000 \). We see from (4) that

\[
Z(n) \equiv -d(n) \pmod{n}
\]

It implies that.

\[
Z(n) + 1 \equiv 1 - d(n) \pmod{n}
\]

By the proof of Theorem 1, we have \( n \mid Z(n)(Z(n) + 1)/2 \), by (2). It can be written as

\[
Z(n)(Z(n) + 1) \equiv 0 \pmod{n}.
\]

Substituting (9) and (10) into (11), we get

\[
d(n)(d(n) - 1) \equiv 0 \pmod{n}.
\]

Notice that \( d(n) > 1 \) if \( n > 1 \). We see from (12) that

\[
(d(n))^2 > n
\]

Let (5) be the factorization of \( n \). By (5), (6) and (13), we obtain

\[
1 > \prod_{i=1}^{r_i} \frac{p_i^{r_i}}{(r_i + 1)^2}
\]

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On the other hand, it is a well known fact that $Z(p^r) \geq p^r - 1 > (r + 1)^2$ for any prime power $p^r$ with $p^r > 32$. We find from (14) that $k \geq 2$.

If $p_1 > 3$, then $p_i^{r_i}/(r_i + 1)^2 \geq 5/4 > 1$ for $i = 1, 2, \cdots k$. It implies that if (14) holds, then either $p_1 = 2$ or $p_1 = 3$. By the same method, then $n$ must satisfy one of the following conditions:

(i) $p_1 = 2, p_2 = 3$ and $(r_1, r_2) = (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (1, 2), (2, 2), (3, 2), (4, 2)$ or $(5, 2)$.

(ii) $p_1 = 2, p_2 > 3$ and $r_1 \leq 5$.

(iii) $p_1 = 3$ and $r_1 = 1$.

However, by (14), we can calculate that $n < 10000$, a contradiction. Thus, the theorem is proved.

References


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Let $p_n$ denote the $n$-th prime number. One of Smarandache's conjectures in [3] is the following inequality:

$$\frac{p_{n+1}}{p_n} \leq \frac{5}{3}, \text{ with equality for } n = 2. \quad (1)$$

Clearly, for $n = 1, 2, 3, 4$ this is true, and for $n = 2$ there is equality. Let $n > 4$. Then we prove that (1) holds true with strict inequality. Indeed, by a result of Dressler, Pigno and Young (see [1] or [2]) we have

$$\frac{p_{n+1}^2}{p_n} \leq 2p_n^2. \quad (2)$$

Thus $\frac{p_{n+1}}{p_n} \leq \sqrt{2} \leq \frac{5}{3}$, since $3\sqrt{2} < 5$ (i.e. $18 < 25$). This finishes the proof of (1).

References:


On the Smarandache Irrationality Conjecture

Florian Luca

The Smarandache Irrationality Conjecture (see [1]) claims:

Conjecture.

Let \( a(n) \) be the \( n \)th term of a Smarandache sequence. Then, the number

\[
0.a(1)a(2)...a(n)...
\]

is irrational.

Here is an immediate proof in the following cases:

1. \( a(n) = n \);
2. \( a(n) = d(n) \) = number of divisors of \( n \);
3. \( a(n) = \omega(n) \) = number of distinct prime divisors of \( n \);
4. \( a(n) = \Omega(n) \) = number of total prime divisors of \( n \) (that is, counted with repetitions);
5. \( a(n) = \phi(n) \) = the Euler function of \( n \);
6. \( a(n) = \sigma(n) \) = the sum of the divisors of \( n \);
7. \( a(n) = p_n \) = the \( n \)th prime;
8. \( a(n) = \pi(n) \) = the number of primes smaller than \( n \);
9. \( a(n) = S(n) \) = the Smarandache function of \( n \);
10. \( a(n) = n! \);
11. \( a(n) = a^n \), where \( a \) is any fixed positive integer coprime to 10 and larger than 1;
12. \( a(n) \) = any fixed non-constant polynomial in one of the above;

Here is the argument:
Assume that

\[
0.a(1)a(2)...a(n)...
\]

is rational. Write it under the form

\[
0.a(1)a(2)...a(n) ... = 0.ABBBB...
\]

where \( A \) is some block of digits and \( B \) is some other repeating block of digits. Assume that \( B \) has length \( t \). If there exist infinitely many \( a(n) \)'s such that the decimal representation of \( a(n) \) contains at least \( 2t \) consecutive zeros, then, since \( B \) has length \( t \), it follows that the block of these \( 2t \) consecutive zeros will contain a full period \( B \). Hence, \( B = 0 \) and the number has, in fact, only finitely many nonzero decimals, which is impossible because \( a(n) \) is never zero.

All it is left to do is to notice that if \( a(n) \) is any of the 12 sequences above, then \( a(n) \) has the property that there exist arbitrarily many consecutive zero's in the decimal representation of \( a(n) \). This is clear for the sequences 1, 2, 3, 4, 8 and 9 because these functions are onto, hence they have all the positive integers in their range. It is also obvious for the sequence 10 because \( n! \) becomes divisible with arbitrarily large powers of 10 when \( n \) is large. For the sequence 7, fix any \( t \) and choose infinitely many primes from the progression \((10^{2t+2k}+1)_{k \geq 0}\) whose first term is 1.
and whose difference is $10^{2t+2}$. This is possible by Dirichlet's theorem. Such a prime will end in ...
$00000001$ with $2t + 1$ consecutive zero's. For the sequence 5, notice that the Euler function of the primes constructed above is of the form $10^{2t+2}k$, hence it ends in $2t + 2$ zeros, while for the sequence 6, notice that the divisor sum of the above primes is of the form $10^{2t+2}k + 2$, hence it ends in ...
$000002$ with $2t + 1$ consecutive zeros. For the sequence 11, since $a$ is coprime to 10, it follows that for any $t$ there exist infinitely many $n$'s such that $a^n \equiv 1 \pmod{10^{2t+2}}$. To see why this happens, simply choose $n$ to be any multiple of the Euler function of $10^{2t+2}$. What the above congruence says, is that $a^n$ is of the form ...
$000001$ with at least $2t + 1$ consecutive zeros (here is why we don't want $a$ to be 1).

Now 12 should also be obvious. It is also clear that the argument can be extended to any base.

It certainly seems much harder to conclude if any one of those series is transcendental or not.

Reference


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A NOTE ON $S(n)$, WHERE $n$ IS AN EVEN PERFECT NUMBER

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In a recent paper [1] the following result is proved:
If $n = 2^{k-1}(2^k-1)$, $2^k-1$ = prime, is an even perfect number, then $S(n) = 2^k-1$, where $S(n)$ is the well-known Smarandache Function.

Since $S(ab) = \max \{ S(a), S(b) \}$ for $(a, b) = 1$, and $S(a) \leq a$ with equality for $a = 1, 4$, and $a = \text{prime}$ (see [3]), we have the following one-line proof of this result:

$S(2^{k-1}(2^k-1)) = \max \{ S(2^{k-1}), S(2^k-1) \} = 2^k-1$,
since $S(2^{k-1}) < 2^k-1$ for $k \geq 2$.

On the other hand, if $2^k-1$ is prime, then we have $S(2^k-1) \equiv 1 \pmod{k}$; an interesting table is considered in [2]. Indeed, if $k$ must be a prime too, $k = p$; while Fermat's little theorem gives $2^p-1 \equiv 1 \pmod{p}$. From $2^{2p}-1 = (2^p-1)(2^p+1)$ and $(2^p-1, 2^p+1) = 1$ we can deduce $S(2^{2p-1}) = \max \{ S(2^p-1), S(2^p+1) \} = 2^p-1$ since $2^p+1$ is being composite, $S(2^p+1) < 2/3(2^p+1) < 2^p-1$ for $p \geq 3$. Thus, if $2^k-1$ is a Mersenne prime, then $S(2^k-1) = S(2^{2p-1}) \equiv 1 \pmod{k}$. If $2^p-1$ and $2^{2p}+1$ are both primes, then

$S(2^{4p}-1) = \max \{ S(2^{2p}-1), S(2^{2p}+1) \} = 2^{2p}+1 \equiv 1 \pmod{4p}$.

References:

Given a positive integer \( n \), let \( P(n) \) denote the largest prime factor of \( n \) and \( S(n) \) denote the smallest integer \( m \) such that \( n \) divides \( m! \).

This paper extends earlier work \([1]\) on the average value of the Smarandache function \( S(n) \) and is based on a recent asymptotic result \([2]\):

\[
\left| \{n \leq N: P(n) < S(n)\} \right| = o\left( \frac{N}{\ln(N)^j} \right)
\]

for any positive integer \( j \) due to Ford. We first prove:

**Theorem 1.**

\[
E(S(N)^k) = \frac{1}{N} \sum_{n=1}^{N} S(n)^k = \frac{\zeta(k+1)}{k+1} \cdot \frac{N^k}{\ln(k)} + O\left( \frac{N^k}{\ln(N)^2} \right)
\]

where \( \zeta(x) \) is the Riemann zeta function. In particular,

\[
\lim_{N \to \infty} \frac{\ln(N)}{N} \cdot E(S(N)) = \frac{\pi^2}{12} = 0.82246703...
\]

\[
\lim_{N \to \infty} \frac{\ln(N)}{N^2} \cdot Var(S(N)) = \frac{\zeta(3)}{3} = 0.40068563...
\]

**Sketch of Proof.**

On one hand,

\[
L(k) = \lim_{n \to \infty} \frac{\ln(n)}{n^k} \cdot E(P(n)^k) \leq \lim_{n \to \infty} \frac{\ln(n)}{n^k} \cdot E(S(n)^k) = \lim_{N \to \infty} \frac{\ln(N)}{N^{k+1}} \cdot \sum_{n=1}^{N} S(n)^k
\]

The above summation, on the other hand, breaks into two parts:

\[
\lim_{N \to \infty} \frac{\ln(N)}{N^{k+1}} \cdot \left( \sum_{P(n)=S(n)} P(n)^k + \sum_{P(n)<S(n)} S(n)^k \right)
\]

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The second part vanishes:

\[
\lim_{N \to \infty} \frac{\ln(N)}{N} \sum_{n=1}^{N} \left( \frac{S(n)}{N} \right)^{k} \leq \lim_{N \to \infty} \frac{\ln(N)}{N} \sum_{n=1}^{N} 1 = \lim_{N \to \infty} \frac{\ln(N)}{N} \cdot \alpha \left( \frac{N}{\ln(N)} \right) = 0
\]

while the first part is bounded from above:

\[
\lim_{N \to \infty} \frac{\ln(N)}{N^{k+1}} \sum_{P(n) \leq S(n)} P(n)^{k} \leq \lim_{N \to \infty} \frac{\ln(N)}{N^{k+1}} \sum_{n=1}^{N} P(n)^{k} = \lim_{n \to \infty} \frac{\ln(n)}{n^{k}} \cdot E(P(n)^{k}) = L(k)
\]

A formula for \( L(k) \) was found by Knuth and Trabb Pardo [3] and the remaining second-order details follow similarly.

Observe that the ratio \( \sqrt{\text{Var}(S(N))} / E(S(N)) \to \infty \) as \( N \to \infty \), which indicates that the traditional sample moments are unsuitable for estimating the probability distribution of \( S(N) \). An alternative estimate involves the relative number of digits in the output of \( S \) per digit in the input. A proof of the following is similar to [1]; the integral formulas were discovered by Shepp and Lloyd [4].

**Theorem 2.**

\[
\lim_{N \to \infty} \mathbb{E} \left( \left\{ \frac{\ln(S(N))}{\ln(N)} \right\}^{k} \right) = \int_{0}^{\infty} x^{k-1} \exp \left( -x - \int_{x}^{\infty} \frac{e^{-y}}{y} \, dy \right) \, dx = \begin{cases} 0.62432998 & \text{if } k = 1 \\ 0.42669576 & \text{if } k = 2 \\ 0.31363067 & \text{if } k = 3 \\ 0.24387660 & \text{if } k = 4 \\ 0.19792289 & \text{if } k = 5 \end{cases}
\]

The mean output of \( S \) hence has asymptotically 62.43% of the number of digits of the input, with a standard deviation of 19.21%. A web-based essay on the Golomb-Dickman constant 0.62432998... appears in [5] and has further extensions and references.

**References**

THE INTANGIBLE ABSOLUTE TRUTH

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In my own work "The Modelling of the Rationality" under the basis of the MESER licence, I have enlightened a new spiritual doctrine sustained by scientific and logical hypotheses.

The reception of the soundness of the mentioned notion proceeds from the Einstein's principle concerning "The internal perfection and its external acknowledgment" but, like other responsible "creators", I felt that it was necessary to consider the expression of the feelings of the uncertainty, mine first.

Although I found many external recognitions in our great forerunners' ideas and theories, we have not had a proven substantiation yet (which is not by all means necessary with philosophical hypotheses) of the hypotheses that I have forwarded.

I am willing to believe it was not accidentally that I got knowledge of the ideas set forth by the mathematician and philosopher Florentin Smarandache, the creator of the Neutrosophy, as a branch of Philosophy, that studies the origin, the character, the aim and the interactions of the neutralities from the spectrum of ideality.

I have established that the Neutrosophy Theory, that belongs to the mentioned thinker, sets up as the scientifically demonstrated fundament for the great majority of the hypotheses I have set forth in "The Modelling of Rationality".

Essentially, Professor Smarandache's Neutrosophy stimulates that for any idea <A> there is also an idea <anti A> and another <neut A>.

The fundamental thesis of the Neutrosophy is: if <A> is t% true and f% false, as bivalent extremes, as a matter of course i% is indeterminant, as a result, t+i+f=100 (or t%+i%+f%=1) which gives a meaning, easily altered, to the usual notions as, for example, the one of complementarity.

Consequently, the complement of t is not f, but i+f, and the complement of f is not t, but t+i.

Florentin Smarandache's theory of Neutrosophy suggests also the fact that any hypothesis has a nature of extreme (it allows an anty-hypothesis and a neutro-hypothesis) which is not bad because t+i+f=100 must be considered dialectically, where both t and f tend to be decreasing without annihilating each other in the advantage of i.

Far from the idea that any hypothesis should not have a nature of extreme, just such a nature is desirable to generate polemics which, in case of confrontation, draws nearer t and f aiming at the neutral equilibrium of the t+f+i=100 relationship, that provides the opportunity of accomplishment.

The theory of Neutrosophy makes obvious the relative nature of the truth and the false, only the neutral nature tending to the absolute owing to its force of accomplishment.
Thanks to the specifications that are stimulated in Smarandache's Neutrosophy, the hypothesis of the MESER concept as: the complementarity between the sacred and the profane, between the divine creation and the intra-specific evolution, the non-contradiction between science and religion, materialism(substantialism) and idealism, between gnosticism and agnosticism, prove to be rational and therefore real and the paradoxes become justified.

Related to the sense of knowledge the MESER concept identifies two modalities: the scientific knowledge that specialises knowledge "more and more from that <<less and less>>" and the philosophic, encyclopaedic knowledge "less and less from that<<more and more>>".

If the first modality is limited especially by the possibilities of communication, the second one is also limited by the insufficient power of comprehensibility of the human mind.

The equilibrium between the two directions which, in the last analysis signifies the way to the truth, is determined by the divine laws of the dissociations, purification (the selection and the dissolving of what is settled, established for good) and those of monadic recomposition, laws that ascertain for the general knowledge a social character, expressed by the syntagme "more and more from <<more and more>>" rendered by the well-known paradox "the more you learn, the less you know."

After all, the fundamental law of Neutrosophy is a successful attempt for resolving the paradox of the knowledge and confirms that the absolute truth is intangible not in a derogatory way but in an optimistic one, approved and significant by the will of God.

Being operative even in the case of the characteristic interpretations, as the present one, Neutrosophy confirms its viability even by the fact that it suggests methods, modalities of evaluations, and new interpretative views.

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ADVANCE OF SMARANDACHE APPROACH TO SOLVING SYSTEMS OF DIOPHANTINE EQUATIONS

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By developing F. Smarandache (algebraic) approach to solving systems of Diophantine equations we elaborate a set of new computative algorithms and analytical formulae, which may be used for finding numerical solutions of some combinatorial and number-theoretic problems.

Key words: systems of Diophantine equations, algebraic approach, combinatorics, number theory, Magic and Latin squares.

1 Introduction

Let it be required to solve some system of Diophantine equations. In this case algebraic methods can be applied for

a) constructing the total algebraic solution of the system;

b) finding the transformations translating an algebraic solution of the system from one form into another one;

c) elucidating the general legitimacies existing between the elements of the algebraic solution;

d) replacing the total algebraic solution containing \( L \) arbitrary selected parameters by a set of algebraic solutions containing less than \( L \) parameters.

This paper is devoted to further advance of algebraic approach to solving systems of Diophantine equations. In particular, in this investigation we

1) describe the simple way of obtaining a total solution of systems of Diophantine linear equations in the integer numbers, and show (see Sect. 2) that this way may be considered as some modification of F. Smarandache algorithm 3 from his work 2;

2) demonstrate the effectiveness of the algebraic approach to the elaboration of computative algorithms and analytical formulae, which may be used respectively for obtaining the required numerical solutions of the discussed systems and for counting of the total quantity of solutions from a given class of numbers (Sect. 3);

3) derive analytical formulae available for constructing classical Magic squares of both odd and even orders (Sect. 4).

2 The way of obtaining the total solution of systems of Diophantine linear equations in the integer numbers

Let it be required to solve some system of linear Diophantine equations in the integer numbers. It seems to be evident, that there is no complication in solving this
problem at present. For instance, one may find in the work\textsuperscript{2} as many as five
different algorithms to obtain a total solution of this problem, which correctness are
proved by mathematical methods and illustrated by concrete examples. In particular,
to illustrate the correctness of an algorithm 3, the system from three following
equations
\[
\begin{align*}
3x_1 + 4x_2 & \quad + 22x_4 \quad - 8x_5 = 25 \\
6x_1 & \quad + 46x_4 \quad - 12x_5 = 2 \\
4x_2 + 3x_3 & \quad - x_4 + 9x_5 = 26
\end{align*}
\]
are solved in the work\textsuperscript{2}. The final solution, obtained for (1) by the mentioned algo­
rithm, has form
\[
\begin{align*}
x_1 &= -40k_1 - 92k_2 + 27; \quad x_2 = 3k_1 + 3k_2 + 4; \quad x_3 = -11k_1 + 8; \\
x_4 &= 6k_1 + 12k_2 - 4; \quad x_5 = 3k_1 - 2,
\end{align*}
\]
where $k_1$ and $k_2$ are any integer numbers.

Let us clear up a question whether (2) is the total solution of system (1) in the
integer numbers. To make it we will solve (2) by the algebraic methods with testing
their correctness on every step of our computations.

1. As well-known\textsuperscript{1-3}, the total algebraic solution of the system (1) may be found
by standard algebraic methods (for instance, by Gauss method). In our case it has the form
\[
\begin{align*}
x_1 &= -(23x_4 - 6x_5 - 1)/3 \\
x_2 &= (x_4 + 2x_5 + 24)/4 \\
x_3 &= (-11x_5 + 2)/3
\end{align*}
\]
that coincides with the solution found on the first step of the algorithm 3 of the
work\textsuperscript{2}.

2. As well-known from the theory of comparison\textsuperscript{3,4}, the total solution in the inte­
ger numbers for the last equation of the system (3) has the form $x_3 = (-11m_1 - 3)/3$, where $m_1$ is any integer number; the value of $m_0$ is equal 0, ±1 or ±2 and
is chosen from the condition that the number $(-11m_0 + 2)/3$ must be integer. Thus,
we find on second step of our computations that $x_3 = 3m_1 + 1$ and $x_3 = -11m_1 - 3$.

We note that the solution $x_5 = 3k_1 - 2$ of (2) may be obtained from our solution
by change of the variable $m_1$ to $k_1 - 1$. Thus, both values of $x_5$ are identical
solutions.

3. Let us get to solving the second equation of the system (3) in the integer numbers.
Replacing the value of $x_5$ by $3m_1 + 1$ in this equation we obtain that
$x_2 = 6 + (6m_1 + x_4 + 2)/4$. Hence it appears (see point 2) that $x_4 = (-2 + 4l_1)m_1 - 2 + 4l_2$,
where $l_1$ and $l_2$ are any integer numbers.

4. Replacing the value of $x_4$ by $(-2 + 4l_1)m_1 - 2 + 4l_2$ in the first equation of the sys­
tem (3) we obtain that $x_1 = 2x_5 - (23(-2 + 4l_1)m_1 + 92l_2 - 47)/3$. Hence it appears (see
points 2 and 3) that $l_1 = 3m_2 + 2$ and $l_2 = 3m_3 + 1$ and, consequently, the total solution
of the system (1) in the integer numbers has the form
\[ x_1 = -4m_1 (2m_2 + 10) - 92m_3 - 13; \quad x_2 = 3m_1 (m_2 + 1) + 3m_3 + 7; \]
\[ x_3 = -11m_4 - 3; \quad x_4 = 6m_1 (2m_2 + 1) + 2(6m_3 + 1); \quad x_5 = 3m_1 + 1, \]

where \( m_1, m_2 \) and \( m_3 \) are any integer numbers.

Comparing (4) with (2) we find that

a) the solution (4) contains greater by one parameter than solution (2);

b) if \( m_2 = 0, m_1 = k_1 - 1 \) and \( m_3 = k_2 \) in the solution (4) then the solution (4) coincides with the solution (2).

Thus, the solution (4) contains all numerical solutions of the system (1), which may be obtained from (2), but a part of numerical solutions, which may be obtained by (4), can not obtain from (2) or, in other words, (2) is not the total solution of the system (1) in the integer numbers.

We add that, in general, a partial loss of numerical solutions of systems of linear Diophantine equations may have more serious consequences than in the discussed case. For instance, as it has been proved in the work by using the algebraic approach to solving systems of Diophantine equations,

*if Magic squares of 4th order contain in its cells 8 even and 8 odd numbers then they can not have structure patterns another than 12 ones, adduced in works* \(^1,3,5-7\) *for Magic squares, contained integer numbers from 1 to 16.*

In reality, this statement is incorrect because yet several new structure patterns may exist for Magic squares from 8 even and 8 odd numbers \(^1,3\).

### 3 Analysing a system from 8 linear Diophantine equations

To demonstrate the effectiveness of the algebraic approach to solving some combinatorial and number-theoretic problems, presented in the form of systems of Diophantine equations, in this section we will analyse the following system from 8 linear Diophantine equations

\[ \begin{align*}
1. \quad & a_1 + a_2 + a_3 = S, \quad 4. \quad & a_1 + a_4 + a_7 = S, \quad 7. \quad & a_1 + a_5 + a_9 = S, \\
2. \quad & a_4 + a_5 + a_6 = S, \quad 5. \quad & a_2 + a_5 + a_8 = S, \quad 8. \quad & a_3 + a_5 + a_7 = S, \\
3. \quad & a_7 + a_8 + a_9 = S, \quad 6. \quad & a_3 + a_6 + a_9 = S, 
\end{align*} \]

We note if symbols \( a_1, a_2, \ldots, a_9 \) are arranged as in the table 1, shown in figure, and their values are replaced by ones, which are taken from some total algebraic solution of the system (5), then table 1 will be transformed into the total algebraic formula of Magic squares of 3rd order. In other words, the discussed problem on solving the system (5) connects direct with the well-know ancient mathematical problem on constructing numerical examples of Magic squares of 3rd order.

#### 3.1 Requirements to a set of numbers, which is the solution of the system (5)

*Proposition 1. A set of nine numbers is a solution of the system (5) only in the case if one succeeds to represent these nine numbers in the form of such three arithmetic
progressions from 3 numbers whose differences are identical and the first terms of all three progressions are also forming an arithmetic progression.

Proof. Using standard algebraic methods (for instance, Gauss method) we find that the total algebraic solution of the system (5) has the form

\[
\begin{align*}
    a_1 &= 2a_5 - a_9; \quad a_2 = 2a_9 + a_6 - 2a_5; \quad a_3 = 3a_5 - a_6 - a_9; \quad a_4 = 2a_5 - a_6; \\
    a_7 &= a_9 + a_6 - a_5; \quad a_8 = 4a_5 - 2a_9 - a_6,
\end{align*}
\]

(6)

where values of parameters \(a_5, a_6\) and \(a_9\) are chosen arbitrarily. Arranging solutions (4) in order, shown in the table 2 (see figure), we obtain the table 3. It is noteworthy that arithmetic progressions with the difference \(2a_5 - a_6 - a_9\) place in the rows of the table 3, whereas ones, having the difference \(a_5 - a_9\), place in its columns. If one introduces three new parameters \(a, b\) and \(c\) by the equalities \(a_5 = a + b + c, \quad a_6 = a + 2c\) and \(a_9 = a + b\) into the table 3, then this table will acquire more elegant form, which it has in table 4, and so the fact of existing of the arithmetic progressions in it will receive more visual impression. Thus, the proof of Proposition 1 follows directly from the construction of tables 3 and/or 4 and it is appeared as a result of using the algebraic methods, mentioned in the points (a) and (c) of Sect. 1.

\[
\begin{array}{ccc}
    a_1 & a_2 & a_3 \\
    a_4 & a_5 & a_6 \\
    a_7 & a_8 & a_9 \\
\end{array}
\]

\[
\begin{array}{ccc}
    2a_9 + a_6 - 2a_5 & a_9 & 2a_5 - a_6 \\
    a_5 + a_6 - a_5 & a_5 & 3a_5 - a_6 - a_9 \\
    a_6 & 2a_5 - a_9 & 4a_5 - 2a_9 - a_6 \\
\end{array}
\]

(1) \hspace{1cm} (3)

\[
\begin{array}{ccc}
    a & a + b & a + 2b \\
    a + c & a + b + c & a + 2b + c \\
    a + 2c & a + b + 2c & a + 2b + 2c \\
\end{array}
\]

(2) \hspace{1cm} (4)

Figure. Elucidating the general legitimacies existing between the elements of the solution (6).

3.2 Elaboration of a universal algorithm for finding all numerical solutions of the system (5) from a given class of numbers

Let it be required to find all numerical solutions of the system (5), which belong to the given class of numbers and has \(a_3 = f\). For elaboration of a universal algorithm, solving this problem, we first write out all possible decompositions of the number \(2f\) in the two summands of the following form

\[
2f = x_1(j) + x_2(j),
\]

(7)

where \(j\) is the number of a decomposition; \(x_1(j)\) and \(x_2(j)\) are two such numbers that \(x_1(j) < x_2(j)\) and both ones belong to the given class of numbers. In a complete set of various decompositions of the kind, we fix only one, having, for instance, the number \(k\). Determine for it the number \(d(k)\)

\[
d(k) = 2f - x_1(j).
\]

(8)
Proposition 2. The desirable numerical solution of the system (5), which contains \( a_5 = f \) and numbers \( x_1(k) \) and \( x_2(k) \) of (7), can be found only in the case, if one succeeds to find, among the remaining numbers of the form \( x_1(k) \), an arithmetic progression from three numbers with the difference \( d(k) \).

Proof. The truth of Proposition 2 follows from the construction of the tables 3 and/or 4, shown in figure.

It is evident also, to obtain a complete set of solutions of the system (5) from the given class of numbers, one should repeat the foregoing actions for all the differences \( d(k) \).

3.3 Deriving an analytical formula for counting the quantity of various solutions of the system (5) from natural numbers

Proposition 3. If \( A(m) \) is the total number of various solutions of the system (5) from natural numbers and \( a_5 = m \) then its value may be computed by the formula

\[
A(m) = 9\lceil m/6 \rceil^2 + 3 \left( m \pmod{6} - 8 \right) \lceil m/6 \rceil + 2 - 2 \left( \left( m \pmod{6} + 5 \right) / 6 \right) + 2 - 2 \left( \left( m \pmod{6} + 5 \right) / 6 \right) + 2 - 2 \left( \left( m \pmod{6} + 5 \right) / 6 \right).
\]

Proof. We first write out all possible decompositions of the number \( 2m \) in two distinct terms.

\[
2m = 1 + (2m - 1), \quad d(1) = m - 1, \quad (j = 1, \quad d(1) = m - 1), \quad (10)
\]

\[
2m = 2 + (2m - 2), \quad d(2) = m - 2, \quad (j = 2, \quad d(2) = m - 2),
\]

\[
2m = m - 2 + (m + 2), \quad d(m - 2) = 2, \quad (j = m - 2, \quad d(m - 2) = 2),
\]

\[
2m = m - 1 + (m + 1), \quad d(m - 1) = 1, \quad (j = m - 1, \quad d(m - 1) = 1).
\]

The problem on counting total number of various solutions of the system (5) with \( a_5 = m \) is now reduced, in accordance with the universal algorithm of Sect. 3.2, to counting a total number of various arithmetic progressions consisting of three numbers, which may be composed from the numbers \( 1, 2, \ldots, m - 2, m - 1 \) and such that the differences in these progressions are respectively equal to \( d(m - 1), d(m - 2), \ldots, d(1) \).

To simplify this new problem we shall deduce a recurrence relation which will link the total numbers of various solutions having \( a_5 = m \) and \( a_5 = m - 1 \). For this aim we decompose all the solutions with \( a_5 = m \) in two groups. The solutions, having number 1, will be attributed to the first group. A total number of such solutions will be denoted by \( A_1(m) \). All the remaining solutions we shall attribute to the second group. We decrease now each number by 1 in all solutions of the second group. After this operation, a lot of the second group solutions will represent by themselves a complete set of various solutions from natural numbers with \( a_5 = m - 1 \). Thus, the following relation

\[
A(m) = A_1(m) + A(m - 1)
\]
is valid or, in other words, if we know a value of \( A(m - 1) \) then for finding the value \( A(m) \) it will be sufficient to count the number of the solutions containing \( a_5 = m \) and the number 1. This new combinatorial problem can be reformulated as the following one:

\[ \text{to find a total number of various arithmetic progressions from three numbers which can be composed from the sequence of numbers 1, 2, ..., } m - 2, m - 1 \text{ and such that the first number of these progressions is number 1 and the differences of the progressions are respectively equal to } \{ d(m - 1), d(m - 2), ..., d(1) \}. \]

It seems to be evident, that a total number of the desired progressions coincides with the maximal difference value of the progression \( D_{\text{max}} \) for which one can still find an arithmetic progression of the required form from the set of numbers 1, 2, ..., \( m - 2, m - 1 \). The value of \( D_{\text{max}} \) can be found from the correlation \( 1 + 2D_{\text{max}} = m - 1 \), whence \( D_{\text{max}} = [(m - 2)/2] \), where square brackets denote the integer part. But in reality this value of \( D_{\text{max}} \) is not always coinciding with the value of \( A_1(m) \): to eliminate this non-coincidence we must decrease the total number of arithmetic progressions by one if numbers \( 1 + d(k) \) or \( 1 + 2d(k) \) coincide with the number \( x_1(k) \) of (7).

Let us determine at which values of \( d(k) \) this coincidence occurs:

\[ 1 + d(k_1) = k_1 = m - d(k_1); \quad 1 + 2d(k_2) = k_2 = m - d(k_2), \]

whence \( d(k_1) = (m - 1)/2 \), and \( d(k_2) = (m - 1)/3 \). If \( d(k_1) = (m - 1)/2 \) the number \( 1 + 2d(k_1) > m - 1 \). Consequently, this case is never fulfilled. The coincidence occurs in the second case if \( m - 1 \) is multiple of 3.

If we decompose all \( m \)-numbers in six classes so that the numbers of the form \( 6k \) will be attributed to the first class and those of the form \( 6k + 1 \) — to the second one and so on, where \( k = 1, 2, ..., \), then for all six classes of the \( m \)-numbers one can write out in the explicit form the values of \( D_{\text{max}} \) and \( A_1(m) \):

\[
\begin{align*}
m = 6k, & \quad D_{\text{max}} = 3k - 1, \quad A_1(6k) = 3k - 1; \\
m = 6k + 1, & \quad D_{\text{max}} = 3k - 1, \quad A_1(6k + 1) = 3k - 2; \\
m = 6k + 2, & \quad D_{\text{max}} = 3k, \quad A_1(6k + 2) = 3k; \\
m = 6k + 3, & \quad D_{\text{max}} = 3k, \quad A_1(6k + 3) = 3k; \\
m = 6k + 4, & \quad D_{\text{max}} = 3k + 1, \quad A_1(6k + 4) = 3k; \\
m = 6k + 5, & \quad D_{\text{max}} = 3k + 1, \quad A_1(6k + 5) = 3k + 1.
\end{align*}
\]

Further we shall need the value of the difference \( \Delta A(k, i) \) of the following form

\[ \Delta A(k, i) = A(6(k + 1) + i) - A(6k + i), \]

where \( i = 0, 1, ..., 5 \). Using (14), (13) and (11) we may find an explicit expressions for \( \Delta A(k, i) \). Let, for instance, \( i = 0 \). Then

\[
\begin{align*}
\Delta A(k, 0) & = A(6(k + 1)) - A(6k) = A_1(6(k + 1)) + A_1(6k + 5) + \\
& + A_1(6k + 4) + A_1(6k + 3)) + A_1(6k + 2) + A_1(6k + 1) = \\
& = (3(k + 1) - 1) + (3k + 1) + 3k + 3k + 3k + (3k - 2) = 18k + 1.
\end{align*}
\]

Remaining values of \( \Delta A(k, i) \) for \( i = 0, 1, ..., 5 \) can be found analogously.
\[ \Delta A(k, i) = 18k + 1 + 3i. \]  

(16)

It is evident, since the \( \Delta A(k, i) \) is a linear function from \( k \), values \( A(6k + i) \) may be obtained from the second degree polynomial

\[ A(6k + i) = b_2(i) k^2 + b_1(i) k + b_0(i), \]  

(17)

where \( b_2(i) = 9, b_1(i) = 3i - 8, b_0(i) = 2 - 2[(i + 5)/6] + [i/5] \); square brackets mean the integer part. Taking into account that \( i = (m \mod 6), \ k = [m/6] \) and \( 6k + i = m \) we may obtain (9) from (17).

It should be noted that using regression analysis methods one may appreciably simplify the expression (9):

\[ A(m) = g((3m^2 - 16m + 18.5)/12), \]  

(18)

where the notation \( g\{a\} \) means the nearest integer to \( a \).

4 Algebraic approach to deriving analytical formulae available for constructing classical Magic squares of the \( n \)-th order

We remind that in the general case1,3 Magic squares represent by themselves numerical or analytical square tables, whose elements satisfy a set of definite basic and additional relations. The basic relations therewith assign some constant property for the elements located in the rows, columns and two main diagonals of a square table, and additional relations, assign additional characteristics for some other sets of its elements. In particular, when the constant property is a significance of sum of various elements in rows, columns or main diagonals of the square, then this square is an Additive one. If an Additive square is composed of successive natural numbers from 1 to \( n^2 \), then it is a Classical one.

It is evident3,10,11 that, from the point of view of mathematics, the analytical solution of the problem on constructing Classical squares of the \( n \)-th order consists of determining a form of \( f \) and \( g \) functions, which permit to compute the position for any natural number \( N \) from 1 to \( n^2 \) in cells of these squares: \( x = f(N, n) \) and \( y = g(N, n) \).

In this section we

1) adduce two types of analytical functions, by which one may construct Classical squares of odd orders;
2) reveal a connection between these analytical functions and Latin squares;
3) give an algebraic generalisation of the notion “Latin square”;
4) derive analytical formulae available for constructing Classical squares of both odd and even orders.

4.1 Classical approach to deriving analytical formulae available for constructing Magic squares of odd order from natural numbers

For any linear algorithmic methods of constructing Classical squares of odd orders, the functions \( f \) and \( g \) have the following forms3,10,11.
\[
\begin{align*}
    f(N, n) &:= a_1(N-1) + b_1[(N-1)/n] + c_1 \pmod{n}, \\
    g(N, n) &:= a_2(N-1) + b_2[(N-1)/n] + c_2 \pmod{n},
\end{align*}
\]

where square brackets mean the integer part; a sign "\(\equiv\)" is the modulo \(n\) equality; \(N\) is any natural number from 1 to \(n^2\); \(a_1, b_1, c_1\) and \(a_2, b_2, c_2\) are such integral coefficients, that the numbers \(a_1, a_2, b_1, b_2; a_1b_2 - a_2b_1; a_2 - a_1, b_2 - b_1, a_2 + a_1\) and \(b_2 + b_1\) are mutually disjoint with \(n\).

There is no difficulty in counting that in formula (19) the coefficients \(\{a_1, b_1, c_1, a_2, b_2, c_2\}\) are equal\(^3,10,11\):

- \(\{1, -1, \lfloor n/2\rfloor; 1, -1, \lfloor n/2\rfloor\}\) for Terrace algorithmic method of constructing Classical squares;
- \(\{1, -1, n/2; 1, -2, n-1\}\) for Siamese method;
- \(\{a = n - 6q\}\) for the classical square of the \(n\)-th order, which, if \(n\) is an odd number, non-divisible by three, can be formed also from a pair of orthogonal Latin squares, constructed by the pair of comparisons \(x + 2y \pmod{n}\) and \(2x + y \pmod{n}\); and so on.

It should be noted, that the above conditions for coefficients of functions \(f\) and \(g\) become contradictory for even \(n\). For example, by the conditions, the coefficients \(a_1, a_2, b_1\) and \(b_2\) of the functions \(f\) and \(g\) of (19) should be mutually disjoint with \(n\), and consequently, if \(n\) is even, they must be odd. The same requirement must be the true for the number \(d = a_1b_2 - a_2b_1\). But if \(a_1, a_2, b_1\) and \(b_2\) are odd, the number \(d\), which is the difference of the two odd numbers, will be an even number.

Thus, an essential fault of linear formulae of (19) is the impossibility of using them for constructing Classical squares of even orders.

4.2 Revealing a connection between Latin squares and analytical formulae of (19)

**Proposition 4.** If a Classical square of the \(n\)-th order is constructed by formulae (19), then it may be constructed also by the formula

\[
N(x, y) = np(x, y) + r(x, y) + 1,
\]

where \(p(x, y) \equiv \alpha_1 x + \beta_1 y + \sigma_1\) and \(r(x, y) \equiv \alpha_2 x + \beta_2 y + \sigma_2\).

**Proof.** The equivalence of formulae (19) and (20) appears from their linearity and the fact, that (20) are inverse formulae to (19). In particular, if values of coefficients \(\{a_1, b_1, c_1, a_2, b_2, c_2\}\) of formulae (19) are known then values of \(\{\alpha_1, \beta_1, \sigma_1, \alpha_2, \beta_2, \sigma_2\}\) of formulae (2) may be computed from following linear equations\(^\text{11}\)

\[
\begin{align*}
    m\alpha_1 &\equiv -a_2; \quad m\beta_1 \equiv a_1; \quad m\sigma_1 \equiv a_2c_1 - a_1c_2; \\
    m\alpha_2 &\equiv b_2; \quad m\beta_2 \equiv -b_1; \quad m\sigma_2 \equiv b_1c_2 - b_2c_1; \\
    m &= a_1b_2 - a_2b_1
\end{align*}
\]

and, reciprocally, at the reverse task, the values of \(\{a_1, b_1, c_1, a_2, b_2, c_2\}\) may be computed from equations:
\[\mu a_1 = -\beta_1; \quad \mu b_1 = \beta_2; \quad \mu c_1 = \beta_1\sigma_2 - \beta_2\sigma_1;\]
\[\mu a_2 = \alpha_1; \quad \mu b_2 = -\alpha_3; \quad \mu c_2 = \alpha_2\sigma_1 - \alpha_1\sigma_2;\]
\[\mu = \alpha_1\beta_2 - \alpha_2\beta_1.\]

For instance, values of \(\{\alpha_1, \beta_1, \sigma_1, \alpha_2, \beta_2, \sigma_2\}\) of formulae (20) are equal

- \(\{(n+1)/2, (n-1)/2, (n-1)/2, (n+1)/2, (n-1)/2, 0\}\) for Terrace method;
- \(\{1, (n-1)/2; 2, (n-1)/2\}\) for Siamese method;
- \(\{(n/2), a, n-a; (n+1)/2, a, a\}\) for Knight method, where \(a = dc + (n-d) (1-c), d = [(n+1)/4], c = [(n \text{ mod } 4)/2].\)

We remind that, a quadratic table \(n \times n\) in size is Latin square of \(n\)-th order if only \(n\) elements of this table are different and each of these \(n\) elements occurs only one time in each row and column of the table. The two Latin squares \(P\) and \(R\) of the same order \(n\) are called orthogonal if all the pairs, formed by their elements \(p_{ij}\) and \(r_{ij}\) (\(i\) is the number of a row; \(j\) is the number of a column) are different.

**Proposition 5.** If elements of a Latin square of the \(n\)-th order are numbers 0, 1, ..., \(n-1\), then, for constructing such Latin square, one may use a linear comparison

\[L(x, y) \equiv \alpha x + \beta y + \sigma,\]

(23)

where \(\alpha\) and \(\beta\) are integer numbers, which are to be mutually disjoint with \(n\); \(\sigma\) is any integer number.

**Proof.** Let the numbers \(L(x, y)\) of (23) are located in each cells of a quadratic table \(n \times n\) in size. We consider \(n\) numbers, which are located in row \(y_0\) of this table. Since the discussed numbers are obtained from the linear comparison (23) at \(x = 0, 1, ..., n - 1\), to show that all they are different, we should demonstrate that they belong to different modulo \(n\) classes. Let \(x_1 > x_2\) and \(\alpha x_1 + \beta y_0 + \sigma = \alpha x_2 + \beta y_0 + \sigma\). Since \(\beta y_0 + \sigma\) is a constant, in accordance with the properties of comparisons \(3, 4\), we obtain the new equality \(\alpha x_1 = \alpha x_2\). Hence, since \(\alpha\) is mutually disjoint with \(n\), \(x_1 \equiv x_2\). But this equality contradicts our assumption. Thus, each of numbers 0, 1, ..., \(n - 1\) occurs only one time in each row and column of the discussed table and so this table is Latin square of \(n\)-th order.

**Proposition 6.** Every Classical square of the odd order, decomposed on two orthogonal Latin squares, may be constructed by the formulae (19) and otherwise.

**Proof.** The truth of Proposition 6 follows directly from Propositions 4 and 5 and conditions for coefficients of functions \(f\) and \(g\) of (19).

4.3 Deriving analytical formulae available for constructing Classical squares of both odd and even orders

**The way 1.** Let us give an algebraic generalisation of the notion “Latin square”:

_a quadratic table \(n \times n\) in size is the generalised Latin square of \(n\)-th order if only \(n\) elements of this table are different and each of these \(n\) elements occurs only \(n\) times in this table._
Proposition 7. Every Classical square of a order \( n \) may be decomposed on two orthogonal generalised Latin squares \( P \) and \( R \) of the order \( n \).

Proof. To prove Proposition 7, it is sufficient to note that

a) any integer number \( N \) from 1 to \( n^2 \) may be presented in the form

\[
N = np + r + 1,
\]

where \( p \) and \( r \) can take values only 0, 1, ..., \( n - 1 \);

b) each of values 0, 1, ..., \( n - 1 \) of parameters \( p \) and \( r \) occurs \( n \) times precisely in the decomposition (24) of numbers \( N \).

Thus, to construct two orthogonal generalised Latin squares \( P \) and \( R \) from a Classical square of a order \( n \), one should replace in the Classical square all numbers \( N \) by respectively \( (N - 1) \mod n \) and \( [(N - 1)/n] \).

Proposition 8. Every Classical square of order \( n \) may be constructed by the formula (20), in which functions \( p(x, y) \) and \( r(x, y) \) may belong, in general case, to both linear and non-linear classes of ones.

Proof. The truth of Proposition 8 follows directly from Propositions 7 and materials of Sect. 4.1.

We note, in particular, one may construct Classical squares of even-even orders \( n \) \( (n = 4k; k = 1, 2, ...) \) by the analytical formula (20), in which functions \( p(x, y) \) and \( r(x, y) \) have the following forms \(^3, ^{11}, ^{13} \)

\[
p(x, y) = cx + (1-c)(n-x-1) \quad \text{and} \quad r(x, y) = (1-c)y + c(n-y-1),
\]

where \( c = \{[(x+1)/2] + [(y+1)/2]\} \mod 2 \); or

\[
p(x, y) = cd - x - 1 + (1-c)(n-d) \quad \text{and} \quad r(x, y) = by + (1-b)(n-y-1),
\]

where \( c = (x + y + a) \mod 2 \); \( d = (1-a)y + a(n-y-1) \); \( b = \{[(x+3)/2] + [y/2] + a\} \mod 2 \); \( a = [2y/n] \); and so on.

The way 2. It is evident, we may consider Classical squares not only as the sum of two orthogonal generalised Latin squares (see the way 1) but, for instance, as quadratic tables whose rows contain certain numerical sequences. Let us look into the problem on finding universal analytical formulae for constructing Classical squares from this new point of view.

Proposition 9. If a Classical square of the \( n \)-th order is constructed by formulae (19), then it may be constructed also by the formula

\[
N(x, y) = a + b - \lambda c,
\]

where \( a, b \) and \( c \) are any integer numbers; \( \lambda \) is 0 or 1, the sign \( "=\)" is the modulo \( n^2 \) equality.

Proof. Let a Classical square of an odd order is constructed by formulae (19). It follows from Proposition 4 that this square may be constructed also by formulae (20). We deduct \( x \)-th element of first row from every \( x \)-th element of all \( y \)-th rows of the Classical square. It is evident that the number
The second summand of (29) may have only two values: \((\beta_2 y) \mod n\) or \(n - (\beta_2 y) \mod n\). Thus, we obtain that numbers of any \(y\)-th row of the reformed Classical square may have only two values. By using the mentioned method of constructing formula (27), we find that parameters of this formula \(a, b, c\) and \(\lambda\) are connected with parameters of the formula (20) by correlations

\[
a = n\{(\alpha_1 x + \sigma_1) \mod n\} + (\alpha_2 x + \sigma_2) \mod n, \\
b = n\{((\alpha_1 x + \sigma_1) + (\alpha_2 x + \sigma_2)) \mod n\}, \\
c = n, \\
\lambda = \frac{1 - \text{sign}((\alpha_2 x + \beta_2 y + (64) \mod n) - (\alpha_2 x + (64) \mod n))}{2},
\]

where \(\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0; \\ 0 & \text{if } x = 0 \end{cases}\) and \(\text{sign}(0) = 0\).

It should be noted, if we get off the sign "\(\equiv\)" in the formula (27) and translate correlations (30) into language of numerical sequences (see the point (c) of Sect. 1), we obtain that, for algorithmic methods which mentioned in Sect. 4.1 and 4.2, the parameters of formula (23) are determined by correlations

\[
a = -(-1)^k n(n-1)/4 + k(n+1)/2 + \{n(n-3)+2\}/4, \\
b = n - 1 - y, \\
c = n, \\
\lambda = \left\lfloor \frac{\text{sign}((h-y) + 2)/2}{2} \right\rfloor,
\]

where the numerical sequence \(\{a_k\}\), if its values are computed at \(k = 0, 1, \ldots, n - 1\), coincides with the numerical sequence, located in the first row of the Classical square; \(\sigma_y(t)\) is a permutation operator of numbers \(0, 1, \ldots, n - 1\) and for

- Terrace method \(k = k_1, z = k_1 - 1;\)
- Siamese method \(z \equiv \{n - 2(k_1 - 1)\} \mod n, k = z + 1;\)
- Knight method \(k_2 = \sigma_y(k_1 - 1), z = -(-1)^k n/4 + k_2/2 + (n+4)/4, k = z + 1.\)

It is evident, using the formula (27) with parameters (31), one can have no difficulty in discovering "genetic connections" between different Classical squares and constructing methods and in generating a set of new methods. For instance, if \(n\) is an odd number, non-divisible by three, the new algorithmic methods for constructing Classical squares of odd orders appear when \(k_1 = \sigma_y^4(x)\) or \(k_1 = \sigma_y^7(x)\) in (31), or the form of \(\sigma_y\) and/or the numerical sequence \(\{a_k\}\) is changed.

It remains for us to add that parameters of the formula (27) are determined by correlations

\[
a = nk, \\
b = w, \\
c = n - 2w - 1, \\
\lambda = [(k+2) \mod 4]/2, \\
\sigma_w(z) = 1 + \{z - h (2 ((z + h) \mod 2) - 1)\} \mod n,
\]
\[ h = y + c\lfloor y/2 \rfloor, \quad k_1 = \sigma_w(z) \]
for formulae (25) and (26), where \( \sigma_w(z) = 1 + (z - h (2((z + h) \mod 2) - 1)) \mod n; \]
\[ h = y + c\lfloor y/2 \rfloor; \quad k_1 = \sigma_w(z) \]
and for
— the formula (26) \( z = x; \quad w = y; \quad k = k_1; \)
— the formula (25) \( \lambda = (((y + 1) \mod 4)/2); \quad k = k_1; \quad \lambda = (((y + 1) \mod 4)/2); \)
\[ z = (x_t + n - h (1 - 2(x_t \mod 2))) \mod n; \quad x_t = \lambda + (1 - \lambda) (n - x_t - 1); \]
\[ w = \lambda y + (1 - \lambda) (n - y - 1). \]

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We discuss the theme on translating different descriptions of computative algorithms into high-level programming languages, enumerate some advantages of analytical descriptions and demonstrate that logical functions may be used effectively to create analytical formulae available for describing a set of combinatorial and number-theoretic computative algorithms. In particular, we adduce analytical formulae to generate $l$-th prime numbers $p_l$, permutations of order $m$, $k$-th numbers of Smarandache sequences of 1st and 2nd kinds and classical Magic squares of an order $n$.

**Key words:** computative algorithms, analytical approach, logical functions, combinatorics, number theory.

1 Introduction

As well-known\(^1,2\) verbal and diagram (graph-diagram) techniques available for describing computative algorithms are the most wide-spread at present. For instance, *Euclidean algorithm*, allowing to find the greatest common divisor (GCD) of the positive integers $a$ and $b$ ($a > b$) has the following verbal description\(^3\)

1. Assign $m = a$, $n = b$;
2. Find $r = m \mod n$;
3. If $r > 0$, then pass to 4. Otherwise, pass to 5;
4. Assign $m = n$, $n = r$ and pass to 2;
5. Answer: $\text{GCD}(a, b) = n$.

Since all computative algorithms are realised, as the rule, on computer at present, the main fault of the verbal description of computative algorithms is the necessity of translating this description into one of special computer-oriented languages.

The diagram form of the description of computative algorithms allows to simplify slightly the process of such translation. In particular, the diagram form of Euclidean algorithm is shown in figure 1, where squares with digits 1, 2, 4, 5 and the rhomb with the condition $r > 0$ mean respectively to points 1, 2, 4, 5 and 3 of the verbal description.

![Figure 1. Diagram form of the description of Euclidean algorithm.](image-url)
The logical technique available for describing computative algorithms is less known than verbal and diagram descriptions, but just it gives easier way to obtain a program code. In particular, we may present the logical description of Euclidean algorithm in the form
\[ A_1 \downarrow^2 A_2 \alpha_3 \uparrow_1 A_4 \beta \uparrow_2 \downarrow^1 A_5, \]
where \( A_i \) are elements of the vector operator \( A = \{(m = a, n = b), (r = m \mod n), (\emptyset), (m = n, n = r), (n)\}; \( \alpha_i \) are elements of the vector conditional jump \( \alpha = \{(\emptyset), (\emptyset), (r > 0), (\emptyset), (\emptyset)\}; \( (\emptyset) \) is the blank in \( A \) and \( \alpha \); \( \uparrow \), and \( \downarrow \) are arrows indicating respectively points of departures and destinations; \( \beta \) is the unconditional jump instruction.

We note that, generally speaking, an one-to-one correspondence exists between three foregoing techniques for describing computative algorithms. In other words these techniques are identical in substance.

One of modern techniques available for describing computative algorithms is using the built-in predicates calculus, realised, for instance, in Prolog. In particular, we may represent Prolog description of Euclidean algorithm by three statements
\[
\text{GCD}(0, V, V).
\text{GCD}(N S, V S, V) :- N S \mod V S = V, \text{GCD}(N S 1, N S, V).
\text{?-GCD}(b, a, V).
\]
Where the second statement is the direct record of the recursive computative procedure, allowing to find \( \text{GCD}(a, b) \); the first one determines the condition to finish this procedure; the third one is constructed to introduce the concrete values of numbers of Euclidean algorithm; \( N S, N S 1, V \) and \( V S \) are internal variables of the procedure and \( V = \text{GCD}(a, b) \) after calculations. The main obstacle of this technique spreading is necessity of preliminary good knowledge of predicates calculus theory.

This paper is devoted to an advance of analytical approach to describing some combinatorial and number-theoretic computative algorithms. Since at present any analytical description of the computative algorithms allows to automate the process of obtaining the program code, we suppose that the discussed theme appears to be interesting.

2 Constructing analytical formulae by using logical functions

2.1 Formulae to generate \( n \)-th prime number \( p_n \)

In our view, the most impressive application of logical functions in elementary number theory is the formula\(^3\)\(^6\) to generate \( n \)-th prime number \( p_n \):
\[
p_n = \sum_{m=0}^{(n+1)^2+1} \text{sg}(n + 1 - \sum_{k=2}^{m} \{(k-1)!^2 - k\{((k-1)!^2 / k)\})
\]
where \( p_0 = 2, p_1 = 3, \ldots \); square brackets mean the integer part; \( \text{sg} \) is a logical function: \( \text{sg}(x) = 1 \) if \( x > 0 \) and \( \text{sg}(x) = 0 \) if \( x \leq 0 \). Let us find another analytical formula for \( p_n \) without factorials.
As well-known\textsuperscript{3, 7}, any prime number has exactly two divisors: the unit and itself. Thus, any integer number \( a \) is a prime one if it has not divisors among integer numbers from 2 to \( \lfloor \sqrt{a} \rfloor \) or, in the language of analytical formulae, if

\[
\chi_a = \prod_{j=2}^{\lfloor \sqrt{a} \rfloor} \left\{ \text{sg}(a - j[i/j]) \right\},
\]

then \( a \) is a prime only when \( \chi_a = 1 \). It appears directly from (4) and (3) that the desirable formula for \( p_n \) has form

\[
p_n = \sum_{m=0}^{(n+1)^2+1} \text{sg}(n-1 - \sum_{a=3}^{m} \chi_a),
\]

where \( p_2 = 2, p_3 = 3, p_4 = 5, \ldots \).

2.2 The analytical description of the permutations generator

As well-known\textsuperscript{2, 3}

a) the permutation of order \( m \) is called any arrangement of \( m \) various objects in a series;

b) the verbal description of the simple algorithm available for constructing all the permutations from \( m \) objects, if all the permutations from \( m - 1 \) objects have been already constructed, has form

Enumerate \( m - 1 \) various objects by the numbers 1, 2, ..., \( m - 1 \). For each permutation of \( a_1, a_2, \ldots, a_{m-1} \), containing the numbers 1, 2, ..., \( m - 1 \), form \( m \) other permutations by putting the number \( m \) in all the possible places. As a result obtain the permutations:

\[
m, a_1, a_2, \ldots, a_{m-1};
\]
\[
a_1, m, a_2, \ldots, a_{m-1};
\]
\[
\ldots \ldots \ldots \ldots \ldots \ldots \ldots ;
\]
\[
a_1, a_2, \ldots, m, a_{m-1};
\]
\[
a_1, a_2, \ldots, a_{m-1}, m.
\]

It is evident that one can obtain all the permutations of order \( m \) by this algorithm and none of the permutations of (6) may be obtained more than once. If this verbal description we translate into one of special computer-oriented languages, for instance, into Pascal, then we obtain the program code, shown in table 1. This program works correctly at initial conditions \( m4 = 1; n1 = m \) and the array \( \text{nb3} \) contains such numbers in the first \( m \) cells, which should be rearranged, and has following advantages over the verbal description

a) the knowledge of all permutations from \( m - 1 \) objects is not required for generating the permutations of order \( m \);

b) permutations are realised with any set of numbers, contained in the first \( m \) cells of the array \( \text{nb3} \).
Table 1. The representation of the permutations generator as Pascal program.

| Procedure Perm(Var m4,n1,n:integer; Var nb3,nb4,nb5:Ten); Label A28,A29,A30; Var n,t,k,m2:Integer; Begin If m4=1 Then Begin m4:=0;n:=n1; For k:=2 to n do Begin nb4[k-1]:=0; nb5[k-1]:=1; End; Exit; End; k:=0; n:=n1; A28: m2:=nb4[n]+nb5[n];nb4[n]:=m2; If m2=n Then Begin nb5[n]:=-1;Goto A29; End; If Abs(m2)>0 Then Goto A30; nb5[n]:=-1;Inc(n); A29: If n>2 Then Begin Dec(n);Goto A28; End; Inc(m2);m4:=1; A30: m2:=m2+k; nt:=nb3[m2]; nb3[m2]:=nb3[m2+1]; nb3[m2+1]:=nt End; |

It should be noted that the main fault of both the verbal description and the program code is the fact that the knowledge of the previous permutation of order \(m\) is required for constructing the next permutation from \(m\) objects. To eliminate this fault one may use a set of analytical formulae:

\[
\begin{align*}
    r_j &= p_j - z_1 + 1, \quad p_j = j - 1 + f(1 - c_j) + c_j (m - j - f), \\
    f &= t_{j-1} - (m - j + 1) \left( t_{j-1} / (m - j + 1) \right), \\
    t_j &= \left[ k / \prod_{q=1}^{j} (m-q+1) \right], \quad c_j = \left| (-1)^{t_j} - 1 \right| / 2, \\
    z_1 &= \text{sg}(1 + p_{j-1} - p_j - z_2) + z_2, \quad z_2 = \text{sg}(1 + p_{j-2} - p_j - z_3) + z_3, \ldots, \\
    z_{j-1} &= \text{sg}(1 + p_1 - p_j),
\end{align*}
\]

where \(k\) is a number of permutation, generated of (7); \(r_j\) is a number, which \(j\)-th element of the initial sequence \(nb3\) has in \(k\)-th permutation; the all another parameters in (7) are auxiliary.

2.3 The formula for counting the value of GCD(a, b)

We may present one of possible formulae available for counting the value of GCD(a, b) \{see Sect. 1\} as

\[
\begin{align*}
    \text{GCD}(a, b) &= b \{1 - \text{sign}(r)\} + k \text{sign}(r), \quad r = a - b[a/b], \\
    k &= \text{MAX}_{i=2}^{|\sqrt{b}|} \{i(1-d)\}, \quad d = \text{sign}\{a - i[a/i]\} \times \text{sign}\{b - i[b/i]\},
\end{align*}
\]

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where the function \( \text{MAX}(a_1, a_2, \ldots, a_r) \) gives the greatest from numbers \( a_1, a_2, \ldots, a_r \); \( \text{sign}(x) = |x|/x \) if \( x \neq 0 \) and \( \text{sign}(0) = 0 \).

2.4 Formulae for the calculation of \( n \)-th numbers in Smarandache sequences of 1st and 2nd kinds

As we found earlier\(^7\), the terms of six Smarandache sequences of 1st kind\(^8\) may be computed by means of one general recurrent expression

\[
a_{\varphi(n)} = \sigma(a_n 10^{\varphi(a_n)} + a_n + 1),
\]

where \( \varphi(n) \) and \( \psi(a_n) \) are some functions; \( \sigma \) is an operator. For instance,

a) if \( \varphi(n) = n + 1, \sigma = 1 \) and \( \psi(a_n) = [\lg(n + 1)] + 1 \) then the following sequence of the numbers, denoted as \( S_1\text{-sequence} \), is generated by (9)

\[
1, 12, 123, 1234, 12345, 123456, \ldots \tag{10}
\]

Let each number \( \chi_k \), determined as

\[
\chi_k = -1 + \sum_{j=0}^{[\lg(k+0.5)]}(k + 1 - 10^j),
\]

(11)

correspond to each number \( a_k \) of sequences (10), where the notation "[\lg(y)]" means integer part from decimal logarithm of \( y \). Using (11) we may construct the following analytical formula for the calculation of \( n \)-th number in the \( S_1 \)-sequence:

\[
a_n = 10^{\chi_n} \sum_{i=1}^{n}(i / 10^{\chi_i});
\]

(12)

b) if \( \varphi(n) = n + 1; \sigma = \gamma \) is the operator of mirror-symmetric extending the number \( a_{[(n+1)/2]} \) of \( S_1 \)-sequence from the right with 1-truncating the reflected number from the left, if \( n \) is the odd number, and without truncating the reflected number, if \( n \) is the even number; \( \psi(a_n) = [\lg([(n+1)/2] + 1)] + 1 \), then the following sequence of the numbers, denoted as \( S_2\text{-sequence} \), is generated by (9)

\[
1, 11, 121, 1221, 12321, 123321, 1234321, \ldots \tag{13}
\]

The analytical formula for the calculation of \( n \)-th number in the \( S_2 \)-sequence has the form

\[
a_n = \sum_{i=1}^{[n/2]} i 10^{\chi_i - [\lg i]} \sum_{i=1}^{[(n+1)/2]} i 10^d,
\]

(14)

where \( d = 1 + \chi_{[(n+1)/2]} + \chi_{[n/2]} - \chi_i \); and so on.

We find recently that the terms of Smarandache sequences of 2nd kind\(^8\) may be computed also by the universal analytical formula \{compare with formulae (12) and (5)}

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\[ a_n = \sum_{m=1}^{U_n} \text{sg}(n+2-b - \sum_{i=b}^{m} \chi_i) \]  

(15)

where \( \chi_i \) are the characteristic numbers for Smarandache sequences of the 2nd kind; \( U_n \) is an up-estimation for the value \( a_n \); \( b \) is a constant. For instance, if

\[ U_n = n^2, \quad b = 2, \quad \chi_i = \text{sg} \left( \sum_{k=1}^{g!} \prod_{q=1}^{\left\lfloor \sqrt{c} \right\rfloor} \text{sg} \left( c - q \left\lfloor \frac{q}{c} \right\rfloor \right) \right), \quad g = \left\lfloor \lg i \right\rfloor + 1, \]  

(16)

c = 10^g \sum_{p=1}^{g} \left\{ \left\lfloor \frac{i}{10^g-p} \right\rfloor - 10 \left\lfloor \frac{i}{10^g-p+1} \right\rfloor \right\} / 10^p ,

the following sequence of Smarandache numbers of 2nd kind is generated by (15)

\[ 1, 2, 3, 5, 7, 11, 13, 14, 16, 17, 19, 20, 23, 29, 30, 31, 32, 34, \ldots \]  

(17)

We note that

a) in formula (16): \( k \) is a number of the permutation, which is generated from
digits of the number \( i \); \( r_j \) is a number, which \( j \)-th digit of the number \( i \) has in \( k \)-th permutation {see (7)};

b) only such integer numbers belong to the sequence (17), which are prime
numbers or can be derived to prime numbers by a permutation of digits in the initial
natural numbers {the number 1 is related to prime numbers in this sequence}.

2.5 Formulae for analytical description of Magic squares constructing methods

As we discovered earlier, logical functions may be used effectively to create
analytical formulae available for describing computational algorithms on constructing
Magic squares of any order \( n \). For instance, let us consider a well-known “Method of
two squares”, whose verbal description has the form:

1. Make a drawing of two square tables of any order \( n = 4k + 2 \) \( (k = 1, 2, \ldots) \).
Divide every table in four equal squares which we shall call A, B, C and D squares
respectively {see figure 2(1)};

2. Place a Magic square of order \( m = 2k + 1 \) in the A, B, C and D squares of the
first table. It is obvious {see figure 2(2)}, the first table will have the same sum of
numbers in its rows, columns and main diagonals;

3. Fill the cells of the second table: all cells of A square are to have zeros; cells
of D square — numbers \( u = m^2 \); cells of B square — numbers \( 2u \) and cells of C
square — numbers \( 3u \). The obtained table {see figure 2(3)} will have the same sum
of numbers only in its columns;

4. Perform such rearrangement of the numbers in the table 2(3) that the new table
will have the same sum of numbers in its rows, columns and main diagonals. It can
be achieved, for instance, by operations

a) underline in the square A of the second table

— \( k \) zeros, located in the extreme left positions of all rows, excepting the middle
row {see figure 2(3)};
- the zero, located in the central cell of the middle row, and another \( k - 1 \) zeros, located left of the central cell.

Exchange all marked zeros against the respective numbers of the square C (see figure 2(4)) and otherwise;

b) mark \( k - 1 \) numbers \( 2u \) in the extreme right positions of every row of square B (see figure 2(3)), and then exchange ones against corresponding numbers of the square D (see figure 2(4)) and otherwise.

5. Add (cell-wise) two auxiliary tables (the Magic square of order 10, obtained as a result of adding auxiliary squares 2(2) and 2(4) is shown in figure 2(5)).

![Image of tables and figures](image-url)

Figure 2. Method of two squares.

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If the foregoing verbal description we translate into Pascal, we obtain the program code, shown in table 2. In this program code

a) the Magic square of order $m = 2k + 1$, located in the A, B, C and D squares of the auxiliary square, may be built for instance, by the functions $3, 11, 13$

$$f(z, m) \equiv (z - 1) + \left[ \frac{(z - 1)}{m} \right] - \left\lfloor \frac{m}{2} \right\rfloor,$$
$$g(z, m) \equiv (z - 1) - \left[ \frac{(z - 1)}{m} \right] + \left\lfloor \frac{m}{2} \right\rfloor,$$

where square brackets mean the integer part; a sign "=" is the modulo $m$ equality; $z$ is any natural number from 1 to $m^2$; functions $f$ and $g$ afford to compute the position of any natural number $z$ in cells of the Magic square: $x = f(z, n)$ and $y = g(z, n)$. In particular, functions (18) may be presented as following two distinct Pascal-procedures

<table>
<thead>
<tr>
<th>Function FX(z:Integer):Integer;</th>
<th>Function gY(z:Integer):Integer;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Begin</td>
<td></td>
</tr>
<tr>
<td>$fx := 1 + (z - 1) + (z - 1) \text{div} n - n \text{ shr} 1$;</td>
<td></td>
</tr>
<tr>
<td>End;</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Function Ind(x,y:Integer):Integer;</th>
<th>Function Sign(n:Word):ShortInt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Begin</td>
<td></td>
</tr>
<tr>
<td>Ind := (x-1)*n+y;</td>
<td></td>
</tr>
<tr>
<td>End;</td>
<td></td>
</tr>
</tbody>
</table>

b) two procedures “Ind” and “Sign” are auxiliary and have the form

<table>
<thead>
<tr>
<th>Function Ind(x,y:Integer):Integer;</th>
<th>Function Sign(n:Word):ShortInt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Begin</td>
<td></td>
</tr>
<tr>
<td>Ind := (x-1)*n+y;</td>
<td></td>
</tr>
<tr>
<td>End;</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Function Sign(n:Word):ShortInt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Begin</td>
</tr>
<tr>
<td>If Odd(n) Then Sign := -1 Else Sign := 1;</td>
</tr>
<tr>
<td>End;</td>
</tr>
</tbody>
</table>
The analytical description of “Method of two squares” has the form

\[ x = i + c_1 m - 1; \quad y = (1 - c_3 - c_6 - c_5) (j + c_2 m) + (c_3 + c_6 + c_5) (1 + ((j + (c_2 + 1)m - 1) \mod n)) - 1; \]

\[ u = m^2; \quad z = 1 + ((N - 1) \mod u); \quad i = f(z, m) + 1; \]

\[ j = g(z, m) + 1; \quad c_1 = (((N - 1)/u) + 1) \mod 4)/2; \]

\[ c_2 = [(N - 1)/u] \mod 2, \quad c_3 = [\text{sign}(c_1 m + i - 3k - 4) + 2)/2]; \]

\[ c_4 = \text{asg}(j - k - 1); \quad c_6 = c_4 [\text{sign}(k - i - c_1 m) + 2)/2]; \]

\[ c_5 = (1 - c_4)(1 - c_1) [\text{sign}(i - 1) + 1]/2 \times [\text{sign}(k - i + 1) + 2)/2], \]

where \( n = 4k + 2 \) is an order of the desirable Magic square, contained natural numbers \( N \) from 1 to \( n^2 \); \( m = 2k + 1 \); functions \( f(z, m) \) and \( g(z, m) \) are determined by the expression (18); \( \text{asg}(x) = 1 \) if \( x \neq 0 \) and \( \text{asg}(0) = 0 \) \( \{\text{asg}(x) = |\text{sign}(x)| = \text{sign} \times |x|\} \).

References

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NUMERICAL FUNCTIONS AND TRIPLETS

I. Bălăcenoiu, D. Bordea, V. Seleacu

We consider the functions: \( f_s, f_d, f_p, F : \mathbb{N}^* \to \mathbb{N} \), where
\[ f_s(k) = n, \quad f_d(k) = n, \quad f_p(k) = n, \quad F(k) = n, \]
\( n \) being, respectively, the least natural number such that \( k/n! - 1, k/n! + 1, k/n! \pm 1 \), or \( k/n! \pm 1 \). This functions have the next properties:

1. Obviosly, from definition of this function, it results:
\[ F(k) = \min\{S(k), f_p(k)\} = \min\{S(k), f_s(k), f_d(k)\} \]

where \( S \) is the Smarandache function (see [3]).

2. \( F(k) \leq S(k), F(k) \leq f_s(k), F(k) \leq f_d(k), F(k) \leq f_p(k) \)

3. \( F(k) = S(k) \) if \( k \) is even, \( k \geq 4 \).

Proof. For any \( n \in \mathbb{N}, n \geq 2 \), \( n! \) is even, \( n! \pm 1 \) are odd. If \( k \) is even, then \( k \) cannot divide \( n! \pm 1 \). So \( F(k) = S(k) = n \geq 2 \) if \( k \) is even, \( k \geq 4 \).

4. If \( p > 3 \) is prime number, then \( F(p) \leq p - 2 \).

Proof. According to Wilson's theorem \((p-1)! + 1 = M_p\). Because \((p-2)! - 1 + (p-1)! + 1 = (p-2)! p \) results for \( p > 3 \), \( (p-2)! - 1 = M_p \) and so \( F(p) \leq p - 2 \).

5. \( F(m!) = F(m! \pm 1) = S(m!) = m \).

6. The equation \( F(k) = F(k + 1) \) has infinitely many solutions, because, according to the property 5), there is the solutions \( k = m! \), \( m \in \mathbb{N}^* \).
7. If \( F(k) = S(k) \) and \( n \) is the least natural number such that \( k/n! \), then \( k \) not divide \( s! \pm 1 \) for \( s < n \).
Let \( k = p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r} \). According to \( S(k) = \max \{ S_p(\alpha_i) \} \), it results that \( S(k) \geq p_h \), where \( p_h = \min \{ p_1, p_2, \ldots, p_r \} \).
If \( k \) not divide \( s! \pm 1 \) for \( s \leq p_h \), then \( k \) not divide \( t! \pm 1 \) for \( t > p_h \).
Consequently, if \( k \) not divide \((n - 1)! \), \( k/n! \) and \( k \) not divide \( s! \pm 1 \) for \( s \leq \min \{ n, p_h \} \), then \( F(k) = S(k) = n \).
Obviously, the numbers \( k = 3t \), \( t \) being odd, \( t \neq 1 \), have \( p_h = 3 \) and they satisfy the condition \( 3t \) not divide \( s! \pm 1 \) for \( s = 1, 2, 3 \).
Therefore, for \( k = 3t \), \( t \) odd, \( t \neq 1 \), \( F(3t) = S(3t) = n \), \( n \) being the least natural number such that \( 3t/n! \).

8. The partition "bai" of the odd numbers.

Let \( A = \{ k \in \mathbb{N} | k \text{ odd and } F(k) = S(k) \} \)
\( B = \{ k \in \mathbb{N} | k \text{ odd and } F(k) < S(k) \} \)

\((A, B)\) is the partition "bai" of the odd numbers.
Into \( A \) there are numbers \( k = 3t \), \( t \) odd, \( t \neq 1 \). Obviously, \( A \) has infinitely many elements.
Into \( B \) there are numbers \( k = t! \pm 1 \) with \( t \geq 3 \), \( t \in \mathbb{N} \). Obviously, \( B \) has infinitely many elements.

**Definition 1** Let \( n \in \mathbb{N}^* \). We called triplet \( \hat{n} \), the set:
\( n - 1, n, n + 1 \).

**Definition 2** Let \( k < n \). The triplets \( \hat{k}, \hat{n} \) are separated if \( k + 1 < n - 1 \), i.e. \( n - k > 2 \).

**Definition 3** The triplets \( \hat{k}, \hat{n} \) are \( l_s \)-relatively prime if \((k - 1, n - 1) = 1, (k + 1, n + 1) \neq 1 \).
For example: \( \hat{6} \) and \( \hat{72} \) are \( l_s \)-relatively prime.

**Definition 4** The triplets \( \hat{k}, \hat{n} \) are \( l_d \)-relatively prime if \((k - 1, n - 1) \neq 1, (k + 1, n + 1) = 1 \).

**Definition 5** The triplets \( \hat{k}, \hat{n} \) are \( l \)-relatively prime if \((k - 1, n - 1) = 1, (k + 1, n + 1) = 1 \).
Definition 6 The triplets $\hat{k}$, $\hat{n}$ are $d$-relatively prime if

$$(k - 1, n + 1) = 1, \ (k + 1, n - 1) = 1.$$  

For example: $\hat{2}$ and $\hat{6}$ are $d$-relatively prime.

Definition 7 Let $k < n$. The triplets $\hat{k}$, $\hat{n}$ are $d_\ast$-relatively prime if

$$(k - 1, n + 1) = 1, \ (k + 1, n - 1) \neq 1.$$  

For example: $\hat{6}$ and $\hat{120}$ are $d_\ast$-relatively prime.

Definition 8 Let $k < n$. The triplets $\hat{k}$, $\hat{n}$ are $d_d$-relatively prime if

$$(k - 1, n + 1) = 1, \ (k + 1, n - 1) = 1.$$  

Example: $\hat{6}$ and $\hat{24}$ are $d_d$-relatively prime.

Definition 9 The triplets $\hat{k}$, $\hat{n}$ are $p$-relatively prime if

$$(k - 1, n - 1) = 1, \ (k - 1, n + 1) = 1, \ (k + 1, n - 1) = 1, \ (k + 1, n + 1) = 1.$$  

Obviously, if $\hat{k}$, $\hat{n}$ are $p$-relatively prime, then they are $l$ and $d$-relatively prime.

For example: $\hat{6}$ and $\hat{120}$ are $p$-relatively prime.

Definition 10 Let $k < n$. The triplets $\hat{k}$, $\hat{n}$ are $F$-relatively prime if

$$(k - 1, n - 1) = 1, \ (k + 1, n - 1) = 1,$$

$$(k - 1, n) = 1, \ (k + 1, n) = 1,$$

$$(k - 1, n + 1) = 1, \ (k + 1, n + 1) = 1.$$  

Definition 11 The triplets $\hat{k}$, $\hat{n}$ are $t$-relatively prime if

$$(k - 1, n - 1) \cdot (k - 1, n) \cdot (k - 1, n + 1) \cdot (k, n - 1) \cdot (k, n) \cdot (k, n + 1), \ (k + 1, n - 1) \cdot (k + 1, n + 1) = 6.$$  

For example: $\hat{2}$ and $\hat{4}$ and $t$-relatively prime.

Definition 12 Let $H \subset \mathbb{N}^*$. The triplet $\hat{n}$, $n \in H$ is, respectively, $l_s, l_d, l, d, d_\ast, d_d, p, F, t$-prime concerned at $H$, if $\forall s \in H, s < n$, the triplets $\hat{s}$, $\hat{n}$ are, respectively, $l_s, l_d, l, d, d_\ast, d_d, p, F, t$-relatively prime.

Let $H = \{n!|n \in \mathbb{N}^*\}$. For the triplets $\hat{m}, \hat{m} \in H$ there are particular properties.

Proposition 1 Let $k < n$. The triplets $\hat{(k!)}, \hat{(n!)}$ are separated if $n > \max\{2, k\}$.  

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Proof. Obviously, \( n! - k! > 2 \) if \( n > 2 \) and \( k < n \), i.e. \( n > \max\{2, k\} \).

**Proposition 2** Let \( n > \max\{2, k\} \) and \( M_{kn} = \{m \in \mathbb{N}|k!+1 < m < n!-1\} \).
If \( k_1 < k_2 \) and \( n_1 > \max\{2, k_1\} \), \( n_2 > \max\{2, k_2\} \), then
\( n_1 - k_1 \leq n_2 - k_2 \Rightarrow \text{card}M_{k_1n_1} < \text{card}M_{k_2n_2} \).

**Proof.** For \( n > k \geq 2 \) it is true that
\[
n! - (n - 1)! > k! - (k - 1)! \tag{1}
\]
Let \( n > k \geq 2, 1 \leq s \leq k \). Using (1) we can write:
\[
(n - s - 1)! - (n - s)! > (k - s - 1)! - (k - s)!
\]
By summing these inequalities, it results:
\[
n! - (n - s)! > k! - (k - s)! \tag{2}
\]
Let \( 2 \leq k_1 < n_1, 2 \leq k_2 < n_2, k_1 < k_2, n_1 - k_1 \leq n_2 - k_2 \). Then \( n_2 - n_1 \geq k_2 - k_1 \geq 1 \) and there is \( n_3 \) such that \( n_2 > n_3 \geq n_1 \) and \( n_2 - n_3 = k_2 - k_1 \).
Using (2) we can write:
\( n_2! - n_3! > k_2! - k_1! \)
Since \( n_3! \geq n_1! \) we have:
\[
n_2! - n_1! > k_2! - k_1! \tag{3}
\]
According to \( \text{card}M_{k_1n_1} = n_1! - 1 - (k_1! + 1) \),
\( \text{card}M_{k_2n_2} = n_2! - 1 - (k_2! + 1) \), it results that:
\[
\text{card}M_{k_2n_2} - \text{card}M_{k_1n_1} = n_2! - n_1! - (k_2! - k_1!)
\]
That is, taking into account (3), \( \text{card}M_{k_1n_1} < \text{card}M_{k_2n_2} \).

**Definition 13** Let \( k < n \). The triplets \((k!)\), \((n!)\) are linked if
\( k! - 1 = n \) or \( k! + 1 = n \).

**Proposition 3** For \( k \in \mathbb{N}^* \) there is \( p \) prime number, such that for any \( s \geq p \) the triplets \((k!)\), \((s!)\) are not \( F \)-relatively prime.
Proof. Obviously, for \( k = 1 \) and \( k = 2 \), the proposition is true. If \( n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r} \) divide \( k! - 1 \) or \( k! + 1 \), then \( p_j > k \geq 3 \), for \( j \in \{1, 2, \ldots, i\} \). Let \( \tilde{n} = p_1 \cdot p_2 \cdots p_i \) and \( p = \max\{p_j\} \). Obviously, \( \tilde{n} \geq 3 \) because \( p > k \geq 3 \), \( \tilde{n}/k! - 1 \) or \( \tilde{n}/k! + 1 \).

For any \( s \geq p \), \( \tilde{n}/s! \) and so, the triplets \((k!),(s!))\) are not \( F \)-relatively prime.

Remark 1 i) Let \( k < n \). If \((k!),(n!))\) are linked, then \( n - k = k! - k \pm 1 \).

If \( 2 < k_1 < n_1 \), \((k_1!))\) with \((n_1!))\) are linked and \( k_2 < n_2 \), \((k_2!))\) with \((n_2!))\) are linked, then \( k_1 < k_2 \Rightarrow n_1 - k_1 < n_2 - k_2 \) and in view of the proposition 2, results \( \text{card} M_{n,n_1} < \text{card} M_{n_2,n_2} \).

ii) There are twin prime numbers with the triplet \((n!))\). For example 5 with 7 are from \((3!))\).

Definition 14 Considering the canonical decomposition of natural numbers \( n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r} \), we define \( \tilde{n} = \{p_1^{a_1}, p_2^{a_2}, \ldots, p_r^{a_r}\} \), \( M = \{n|n \in N^*\} \).

Definition 15 On \( M \) we consider the relation of order \( \sqsubseteq \) defined by:

\[ \{p_1^{a_1}, p_2^{a_2}, \ldots, p_r^{a_r}\} \sqsubseteq \{q_1^{b_1}, q_2^{b_2}, \ldots, q_t^{c_t}\} \]

if and only if \( \{p_1, p_2, \ldots, p_r\} \subset \{q_1, q_2, \ldots, q_t\} \) and if \( p_i = q_j \), then \( \alpha_i \leq \beta_j \).

Remark 2 For any triplet \((n!))\), \( n \in N^* \), we consider the sets:

\( A_n = \{k \in N^*|k \sqsubseteq \tilde{n}\} \), \( A_n^* = \{k \in A_n|k \not\in A_h \text{ for } h < n\} \)
\( B_n = \{k \in N^*|k \sqsubseteq n! - 1\} \), \( B_n^* = \{k \in B_n|k \not\in B_h \text{ for } h < n\} \)
\( C_n = \{k \in N^*|k \sqsubseteq n! + 1\} \), \( C_n^* = \{k \in C_n|k \not\in C_h \text{ for } h < n\} \)
\( M_n = \{k \in N^*|k \sqsubseteq \text{card} M_h \text{ for } h < n\} \)
\( M_n^* = \{k \in M_n|k \not\in M_h \text{ for } h < n\} \).

It is obvious that:
\( A_n^* = S^{-1}(n), B_n^* = f_s^{-1}(n), C_n^* = f_d^{-1}(n), M_n^* = F^{-1}(n) \).

If \( k \in A_n^* \), it is said that \( k \) has a factorial signature which is equivalent with the factorial signature of \( n! \) (see [1]).

Let \( k \in B_n^* \), \( k = t_1^{s_1} \cdot t_2^{s_2} \cdots t_i^{s_i} \). Then \( \{t_r\} \not\subseteq \tilde{n}! \text{ for } r = 1, 1 \text{ and for any } h < n, \text{ there are } t_j^* \in \tilde{t}_j \text{ for } 1 \leq j \leq i \), such that \( \{t_j^*\} \not\subseteq h! - 1 \).

Similarly, for \( k \in C_n^* \): \( \{t_r\} \not\subseteq \tilde{n}! \text{ for } r = 1, 1 \text{ and for any } h < n, \text{ there are } t_j^* \in \tilde{t}_j \text{ for } 1 \leq j \leq i \), such that \( \{t_j^*\} \not\subseteq h! + 1 \).
References


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ABSTRACT: In this article I have defined a number of SMARANDACHE type sets, sequences which I found very interesting. The problems and conjectures proposed would give food for thought and would pave ways for more work in this field.

(1) SMARANDACHE PATTERNED PERFECT SQUARE SEQUENCES.
Consider following sequence of numbers
13, 133, 1333, 13333, . . .--------(1)

The sequence formed by the square of the numbers is

169, 17689, 1776889, 177768889, . . .--------(2)

We define (1) as the root sequence
It is evident that the above sequence (2) follows a pattern.
i.e. The square of one followed by n three's is, one followed by (n-1) seven's, followed by a six, followed by (n-1) eight's followed by a nine.
There are a finite number of such patterned perfect square sequences. Here we list the root sequences.

(I) 13, 133, 1333, 13333, . . .
(2) 16, 166, 1666, 16666, . . .
(3) 19, 199, 1999, 19999, . . .
(4) 23, 233, 2333, 23333, . . .
(5) 26, 266, 2666, 26666, . . .
(6) 29, 299, 2999, 29999, . . .

on similar lines we have the root sequences with the first terms as

There are some root sequences which start with a three digit number, like
799, 7999, 79999, . . .
The patterned perfect square sequence is

638401, 63984001, 6399840001, 639998400001, . . .

( the nine's and zero's inserted are shown in darker print to identify the pattern.)
Open Problem: (1) Are there any patterned perfect cube sequences?
(2) Are there any patterned higher perfect power sequences?

(2) SMARANDACHE BREAKUP SQUARE SEQUENCES

4, 9, 284, 61209, ...

the terms are such that we have
4 = 2²
49 = 7²
49284 = 222²
4928461209 = 70203²
Tₙ = the smallest number whose digits when placed adjacent to other terms of the sequence in the following manner

\[ T_1T_2...T_{n-1}T_n \quad \text{yields a perfect square.} \]

\[ \lim_{n \to \infty} \left( \frac{T_1T_2...T_{n-1}T_n}{10^k} \right)^{1/2} \]

where \( k \) is the number of digits in the numerator for this kind of sequence can be analyzed. As it is evident that for large values of \( n \) the value of \( \left( \frac{T_1T_2...T_{n-1}T_n}{10^k} \right)^{1/2} \) is close to either 2.22... or to 7.0203...

(3) SMARANDACHE BREAKUP CUBE SEQUENCES

On similar lines SMARANDACHE BREAKUP CUBE SEQUENCES can be defined. The same idea can be extended to define SMARANDACHE BREAKUP PERFECT POWER SEQUENCES

(4) SMARANDACHE BREAKUP INCREMENTED PERFECT POWER SEQUENCES

1, 6, 6375,

1 = 1⁺¹, 16 = 4², 166375 = 55³, etc.

Tₙ = the smallest number whose digits when placed adjacent to other terms of the sequence in the following manner

\[ T_1T_2...T_{n-1}T_n \quad \text{yields a perfect } n^{\text{th}} \text{ power.} \]
(5) SMARANDACHE BREAKUP PRIME SEQUENCE

2, 3, 3, ... 

2, 23, 233 etc. are primes.

\[ T_1T_2...T_{n-1}T_n \] is a prime

(6) SMARANDACHE SYMMETRIC PERFECT SQUARE SEQUENCE

1, 4, 9, 121, 484, 14641, ...

(7) SMARANDACHE SYMMETRIC PERFECT CUBE SEQUENCE

1, 8, 343, 1331, ...

This can be extended to define

(8) SMARANDACHE SYMMETRIC PERFECT POWER SEQUENCE

(9) SMARANDACHE DIVISIBLE BY \( n \) SEQUENCE

1, 2, 3, 2, 5, 2, 5, 6, 1, 0, 8, 4...

the terms are the smallest numbers such that \( n \) divides \( T_1T_2...T_{n-1}T_n \) the terms placed adjacent digit wise.

E.g. 1 divides 1, 2 divides 12, 3 divides 123, 4 divides 1232, 5 divides 12325, 6 divides 123252, 7 divides 1232535, 8 divides 12325256, 9 divides 123252561, 10 divides 1232525610, 11 divides 12325256108, 12 divides 123252561084, etc.

(9) SMARANDACHE SEQUENCE OF NUMBERS WITH SUM OF THE DIGIT'S = PRIME

2, 3, 5, 7, 11, 12, 14, 16, 20, 21, 23, 25, 29, ...

(10) SMARANDACHE SEQUENCE OF PRIMES WITH SUM OF THE DIGIT'S = PRIME

2, 3, 5, 7, 11, 23, 29, 41, 43, 47, 61, 67, 83, 89, ...

(11) SMARANDACHE SEQUENCE OF PRIMES SUCH THAT 2P + 1 IS ALSO A PRIME

2, 3, 5, 11, 23, 29, 41, 53, ...

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(11) SMARANDACHE SEQUENCE OF PRIMES SUCH THAT 2P - 1 IS ALSO A PRIME
3, 7, 19, 31, . . .

(13) SMARANDACHE SEQUENCE OF PRIMES SUCH THAT P^2 + 2 IS ALSO A PRIME
3, 17, . . .

(14) SMARANDACHE SEQUENCE OF SMALLEST PRIME WHICH DIFFER BY 2n FROM ITS PREDECESSOR
5, 17, 29, 97, . . .
(T_1 = 5 = 3+ 2, T_2 = 17 = 13 + 4, T_3 = 29 = 23 + 6, T_4 = 97 = 89 + 8 etc.)

(15) SMARANDACHE SEQUENCE OF SMALLEST PRIME p FOR WHICH p + 2r IS A PRIME
3, 13, 23, 89, . . .
3 + 2 X 1 = 5 is a prime, 13 + 2 X 2 = 17 is a prime, 23 + 2 X 3 = 29, 89 + 2 X 4 = 97 is a prime . . . etc.

(16) SMARANDACHE SEQUENCE OF THE SMALLEST NUMBER WHOSE SUM OF DIGITS IS n .
1, 2, 3, 4, 5, 6, 7, 8, 9, 19, 29, 39, 49, 59, 69, 79, 89, 99, 199, 299, 399, 499, 599, 699, . . .

It is a sequence of the only numbers which have the following property.

\[ N + 1 = \prod_{r=1}^{k} ( a_r + 1 ) \]

PROOF:
Let N be a k-digit number with a_r the r^{th} digit (a_1 = LSB) such that

\[ N + 1 = \prod_{r=1}^{k} ( a_r + 1 ) \]  \hspace{1cm} \text{----------(1)}

to find all such k-digit numbers.
The largest \( k \)-digit number is \( N = 10^k - 1 \), with all the digits as 9. It can be verified that this is a solution. Are there other solutions?

Let the \( m \)th digit be changed from 9 to \( a_m \) (\( a_m < 9 \)). Then the right member of (1) becomes \( 10^{(k-1)} (a_m + 1) \). This amounts to the reduction in value by \( 10^{(k-1)}(9-a_m) \). The value of the \( k \)-digit number \( N \) goes down by \( 10^{(m-1)}(9-a_m) \). For the new number to be a solution these two values have to be equal which occurs only at \( m = k \). This gives 8 more solutions. In all there are 9 solutions given by \( a \cdot 10^k - 1 \), for \( a = 1 \) to 9.

E.g. for \( k = 3 \) the solutions are

199, 299, 399, 499, 599, 699, 799, 899, 999, ...

Are there infinitely many primes in this sequence.

(17) SMARANDACHE SEQUENCE OF NUMBERS SUCH THAT THE SUM OF THE DIGITS DIVIDES \( n \)

1, 3, 6, 9, 10, 12, 18, 20, 21, 24, 27, 30, 36, 40, 42, 45, 48, 50, 54, 60, 63, 72, 80, 81, 84, 90, 100, 102, 108, 110, 112, 114, 120, 126, 132, 133, 135, 140, 144, 150, ...

(18) SMARANDACHE SEQUENCE OF NUMBERS SUCH THAT EACH DIGIT DIVIDES \( n \)

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 20, 22, 24, 30, 33, 36, 40, 44, 50, 55, 60, 66, ...

(19) SMARANDACHE POWER STACK SEQUENCE FOR \( n \)

**SPSS(2)**

1, 12, 124, 1248, 124816, 12481632.

The \( n \)th term is obtained by placing the digits of the powers of 2 starting from \( 2^0 \) to \( 2^n \) from left to right.

**SPSS(3)**

1, 13, 139, 13927, 1392781, 1392781243, ...

Problem: If \( n \) is an odd number not divisible by 5 how many of the above sequence \( SPSS(n) \) are prime? (It is evident that \( n \) divides \( T_n \) iff \( n = 0 \mod (5) \)).

(20) SMARANDACHE SELF POWER STACK SEQUENCE

**SSPSS**
1, 14, 142, 1427, 1427256, 14272563125, 142725631257776, ...

\[ T_r = T_{r-1}a_1a_2a_3\ldots a_k \text{ where } r^f = a_1a_2a_3\ldots a_k \] (the digits are placed adjacent).

How many terms of the above sequence, SSPSS are prime?

(21) SMARANDACHE PERFECT SQUARE COUNT PARTITION SEQUENCE

The \( r \)th term of \( \text{SPSCPS}(n) \) is defined as

\[ T_r = O\{ x \mid x \text{ is a perfect square}, nr + 1 \leq x \leq nr + n \} \]

\( O \) stands for the order of the set
e.g. for \( n = 12 \) \( \text{SPSCPS}(12) \) is

3, 1, 2, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1

(number of perfect squares \( \leq 12 \) is 3 (1, 4, and 9), number of perfect squares between 13 to 24 is 1 (only 16) etc.)

(21) SMARANDACHE PERFECT POWER COUNT PARTITION SEQUENCE

The \( r \)th term of \( \text{SPPCPS}(n,k) \) is defined as

\[ T_r = O\{ x \mid x \text{ is a } k\text{th perfect power}, nr + 1 \leq x \leq nr + n \} \]

where \( O \) stands for the order of the set

By this definition we get

\( \text{SPSCPS}(12) = \text{SPPCPS}(12,2) \)

Another example, \( \text{SPPCPS}(100,3) \) is

4,1,1,1,0,1,0,1,0,1,0,0,0,1,0,0,0,1,0,0,0,1,0,0,0,1,0,0,0,1,0,0,0,1,...

Problem: Does \( \sum (T_r/(nr)) \) converge as \( n \to \infty \)?

(22) SMARANDACHE BERTRAND PRIME SEQUENCE

According to Bertrand's postulate there exists a prime between \( n \) and \( 2n \). Starting from 2 let us form a sequence by taking the largest prime less than double of the previous prime in the sequence. We get
2, 3, 5, 7, 13, 23, 43, 83, 163, ... 

(23) SMARANDACHE SEMI-PERFECT NUMBER SEQUENCE

6, 12, 18, 20, 24, 30, 36, 40, ... 

A semi perfect number is defined as one which can be expressed as the sum of its (all or fewer) distinct divisors.

\[ \begin{align*}
12 &= 2^2 + 4 + 6 = 1 + 2 + 3 + 6 \\
20 &= 2^2 + 4 + 5 + 10 \\
30 &= 2 + 3 + 10 + 15 = 1 + 3 + 5 + 6 + 15 
\end{align*} \]

It is evident that every perfect number is also a semi perfect number.

**THEOREM**: There are infinitely many semi perfect numbers.

**Proof**: We shall prove that \( N = 2^n p \) where \( p \) is a prime less than \( 2^{n+1} - 1 \), is a semi-perfect number.

The divisors of \( N \) are

\[
\begin{align*}
\text{row 1} & \quad 1, 2, 2^2, 2^3, 2^4, \ldots 2^n \\
\text{row 2} & \quad p, 2p, 2^2p, 2^3p, 2^4p, \ldots 2^np
\end{align*}
\]

we have \( \sum_{r=0}^{n-1} 2^r p = p \left( 1 + 2 + 2^2 + 2^3 + \ldots 2^{n-1} \right) = p(2^n - 1) = M \)

\( M \) is short of \( N \) by \( p \). The task ahead is to express \( p \) as the sum of divisors from the first row. It is an established fact that every number can be expressed as the sum of powers of 2, i.e.

\[ p = \sum_{r=0}^{n} a_r \cdot 2^r, \quad \text{where } a_r = 0 \text{ or } a_r = 1. \text{ iff } p \leq 2^{n+1} - 1, \text{ the equality giving a perfect number.} \]

(note: \( a_1a_2a_3 \ldots a_n \) is the binary representation of \( p \)).

\( N = M + p \) is expressible as the sum of its divisors.

**Remark**: This of-course is not exhaustive. There are many more such examples possible giving infinitely many semi perfect numbers. One can explore the possibility of more such expressions.
**SMARANDACHE CO-PRIME BUT NO PRIME SEQUENCE**

4, 9, 10, 21, 22, 25, 26, 27, 28, 33, 34, 35, 36, 49, 50, 51, 52, ...

The \( n \)th term \( T_n \) is defined as follows

\[ T_n = \{ x \mid (T_{n-1}, x) = 1, \text{x is not a prime and } (T_{n-1}, y) \neq 1 \text{ for } T_{n-1} < y < x \} \]

The smallest number which is not a prime but is relatively prime to the previous term in the sequence.

**Open problem**: Is it possible to as large as we want but finite increasing sequence \( k, k+1, k+2, k+3, \ldots \) included in the above sequence?

**DEFINITION**: We define a prime to be weak, strong or balanced prime accordingly as \( p_r \leq or > \frac{p_{r-1} + p_{r+1}}{2} \). where \( p_r \) is the \( r \)th prime.

e.g. \( 3 < \frac{2+5}{2} \) 3 is weak prime. \( 5 = \frac{3 + 7}{2} \) is a balanced prime. \( 71 > \frac{67 + 73}{2} \) is a strong prime.

**SMARANDACHE WEEK PRIME SEQUENCE**:

3, 7, 13, 19, 23, 29, 31, 37, ...

**SMARANDACHE STRONG PRIME SEQUENCE**:

11, 17, 41, ...

**SMARANDACHE BALANCED PRIME SEQUENCE**:

5, 157, 173, 257, 263, 373, ...

It is evident that for a balanced prime > 5, \( p_r = p_{r-1} + 6k \).

**OPEN PROBLEM**: Are there infinitely many terms in the SMARANDACHE BALANCED PRIME SEQUENCE?
How big is $N$? One of the first estimates of its size was approximately [6]:

$$10^{6846168}$$

But this is a rather large number; to test all odd numbers up to this limit would take more time and computer power than we have. Recent work has improved the estimate of $N$. In 1989 J.R. Chen and T. Wang computed $N$ to be approximately [7]:

$$10^{43000}$$

This new value for $N$ is much smaller than the previous one, and suggests that some day soon we will be able to test all odd numbers up to this limit to see if they can be written as the sum of three primes.

Anyway assuming the truth of the generalized Riemann hypothesis [5], the number $N$ has been reduced to $10^{20}$ by Zinoviev [9], Saouter [10] and Deshouillers. Effinger, te Riele and Zinoviev[11] have now successfully reduced $N$ to 5.

Therefore the weak Goldbach conjecture is true, subject to the truth of the generalized Riemann hypothesis.

Let's now analyse the generalizations of Goldbach conjectures reported in [3] and [4]; six different conjectures for odd numbers and four conjectures for even numbers have been formulated. We will consider only the conjectures 1, 4 and 5 for the odd numbers and the conjectures 1, 2 and 3 for the even ones.

4.1 First Smarandache Goldbach conjecture on even numbers.

Every even integer $n$ can be written as the difference of two odd primes, that is $n = p - q$ with $p$ and $q$ two primes.

This conjecture is equivalent to:

For each even integer $n$, we can find a prime $q$ such that the sum of $n$ and $q$ is itself a prime $p$.

A program in Ubasic language to check this conjecture has been written.
for SLES and SLOS.

(2) SMARANDACHE DIVISOR SEQUENCES:

Define \( A_n = \{ x | d(x) = n \} \)

Then
- \( A_1 = \{ 1 \} \)
- \( A_2 = \{ p | p \text{ is a prime} \} \)
- \( A_3 = \{ x | x = p^2, p \text{ is a prime} \} \)
- \( A_4 = \{ x | x = p^3 \text{ or } x = p_1 p_2, p, p_1, p_2 \text{ are primes} \} \).

\( A_4 \rightarrow 6, 8, 10, 14, 15, 21, 22, 26, 27, \ldots \)

We have

\[ \sum \frac{1}{T_n} = 1 \text{ for } A_1 \]

This limit does not exist for \( A_2 \).

\[ \lim_{n \to \infty} \sum \frac{1}{T_n} \text{ exists and is less than } \pi^2/6 \text{ for } A_3 \text{ as } \lim_{n \to \infty} \sum \frac{1}{n^2} = \pi^2/6. \]

The above limit does exist for \( A_p \) where \( p \) is a prime.

* Whether these limits exist for \( A_4, A_5 \) etc is to be explored.

DIVISOR SUB SEQUENCES

The sub sequences for \( A_4, A_5 \) etc can be defined as follows:

\( B( r_1, r_2, r_3, \ldots, r_k ) \) is the sequence of numbers \( r_1, r_2, r_3, \ldots, r_k \)
in increasing order, where \( p_1, p_2, p_3, \ldots, p_k \) are primes. All the numbers having the same unique factorization structure.

DIVISOR MULTIPLE SEQUENCE

\( SDMS = \{ n | n = k \cdot d(n) \} \).

\( SDMS \rightarrow 1, 2, 8, 9, 12, \ldots \)

(3) SMARANDACHE QUAD PRIME SEQUENCE GENERATOR:

\( SQPSG = \{ r | 90r+11, 90r+13, 90r+17, 90r+19 \text{ are all primes} \} \)
SQPSG $\rightarrow$ 0, 1, 2, …

Are there infinitely many terms in the above sequence?

(4) SMARANDACHE PRIME LOCATION SEQUENCES

Define $P_0 =$ sequence of primes.

$P_1 =$ sequence of primeth primes

$P_1 \rightarrow 3, 5, 11, 17, \ldots$

$P_2 =$ sequence of primeth, primeth prime.

$\downarrow$

$\downarrow$

$P_r =$ sequence of primeth, primeth, ... $r$ times, primes

* If $T_n$ is the $n^{th}$ term of $P_r$, then what is the minimum value of $r$ for which

$$\lim_{n \to \infty} \sum 1/T_n \text{ exists?}.$$ 

(5) SMARANDACHE PARTITION SEQUENCES

(i) PRIME PARTITION

Number of partitions into prime parts

$Sp_p(n) \rightarrow 0, 1, 1, 1, 2, 2, 3, \ldots$

(ii) COMPOSITE PARTITION

Number of partitions into composite parts

$Sp_c(n) \rightarrow 1, 1, 2, 1, 3, \ldots$

(iii) DIVISOR PARTITIONS

Number of partitions into parts which are the divisors of $n$.

$Sp_d(n) \rightarrow 1, 1, 1, 2, 1,$
On similar lines following two partition sequences can be defined.

(iv) CO-PRIME PARTITIONS : \(SP_{cp}(n)\)
Number of partitions into co-prime parts.

(v) NON- CO-PRIME PARTITIONS \(SP_{ncp}(n)\)
Number of partitions into non coprime parts.

(vi) PRIME SQUARE PARTITIONS
Partitions into prime square parts.

This idea could be generalized to define more such functions.

(6) SMARANDACHE COMBINATORIAL SEQUENCES.

(I) Let the first two terms of a sequence be 1 & 2. The \((n+1)^{th}\)
term is defined as
\[ T_{n+1} = \text{sum of all the products of the previous terms of the} \]
sequence taking two at a time.
\[ T_1 = 1, T_2 = 2, \Rightarrow T_3 = 2, \text{ and } T_4 = 8, \]

SCS(2) = 1, 2, 2, 8, 48, \ldots

The above definition can be generalized as follows:
Let \( T_k = k \) for \( k = 1 \) to \( n \).

\[ T_{n+1} = \text{sum of all the products of the previous terms of the} \]
sequence taking \( r \) at a time. This defines \( SCS(r) \).

Another generalization could be:
Let \( T_k = k \) for \( k = 1 \) to \( n \).

\[ T_r = \text{sum of all products of } (r-1) \text{ terms of the sequence taking} \]
\( (r-2) \) at a time \( (r > n) \). This defines \( SC_v S \).

for \( n = 2 \) \( T_1 = 1, T_2 = 2, T_3 = 3, T_4 = 17 \) etc

\( SC_v S \rightarrow 1, 2, 3, 17, \ldots \)
PROBLEM: (1) How many of the consecutive terms of SCS(r) are pairwise coprime?

(2) How many of the terms of SCvS are primes?

(ii) SMARANDACHE PRIME PRODUCT SEQUENCES

SPPS(n)

\[ T_n = \text{sum of all the products of primes chosen from first } n \text{ primes taking } (n-1) \text{ primes at a time.} \]

SPPS(n) → 1, 5, 31, 247, 2927 . . .

\[ T_1 = 1, T_2 = 2 + 3, T_3 = 2 \times 3 + 2 \times 5 + 3 \times 5 = 31. \]

\[ T_4 = 2 \times 3 \times 5 + 2 \times 3 \times 7 + 2 \times 5 \times 7 + 3 \times 5 \times 7 = 247 \text{ etc.} \]

How many of these are primes?

(7) SMARANDACHE \( \phi \)-SEQUENCE

\( S\phi S = \{ n \mid n = k \times \phi(n) \} \)

\( S\phi S \rightarrow 1, 2, 4, 6, 8, 12 . . . \)

(8) SMARANDACHE PRIME DIVISIBILITY SEQUENCE

SPDS = \{ n \mid n \text{ divides } p_n + 1, \text{ } p_n \text{ is the } n^{th} \text{ prime.} \}

SPDS→ 1, 2, 3, 4, 10, . . .
ON THE DIVISORS OF SMARANDACHE UNARY SEQUENCE

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ABSTRACT: Smarandache Unary Sequence is defined as follows:
\[ u(n) = 1111 \ldots, \text{p}_n \text{ digits of "1"}, \text{ where } \text{p}_n \text{ is the n}^{\text{th}} \text{ prime.} \]
11, 111, 11111, 1111111 \ldots
Are there an infinite number of primes in this sequence? It is still an unsolved problem. The following property of a divisor of \( u(n) \) is established.
If 'd' is a divisor of \( u(n) \) then \( d \equiv 1 \pmod{\text{p}_n} \), for all \( n > 3 \) \( \text{---(1)}, \)

DESCRIPTION: Let \( I(m) = 1111, \ldots m \text{ times} = (10^m - 1)/9 \)
Then \( u(n) = I(\text{p}_n) \).
Following proposition will be applied to establish (1).
Proposition : \( I(p - 1) \equiv 0 \pmod{p} \). \( \text{----(2)} \)
PROOF: \( 9 \text{ divides } 10^{p-1} - 1 \). From Fermat's little theorem if \( p \geq 7 \) is a prime
then \( p \text{ divides } (10^{p-1} - 1)/9 \)
as \( (p, 9) = (p, 10) = 1 \). Hence \( p \text{ divides } I(p-1) \)

Coming back to the main proposition, let 'd' be a divisor of \( u(n) \).
Let \( d = p^aq^br^c \ldots \), where \( p, q, r \), are prime factors of \( d \).
p divides \( d \Rightarrow p \text{ divides } u(n) \) also \( p \text{ divides } I(p-1) \) from proposition (2). in other words
\( p \text{ divides } (10^{p-1} - 1)/9 \) and \( p \text{ divides } (10^p - 1)/9 \)
\( p \text{ divides } (10^{A(p-1)} - 1)/9 \) and \( p \text{ divides } (10^{B,p} - 1)/9 \)
\( p \text{ divides } (10^{(A(p-1) - B,p )}/9 \)
\( p \text{ divides } 10^{B,p} \{ (10^{A(p-1) - B,p} - 1)/9 \} \).
\( p \text{ divides } (10^{A(p-1) - B,p} - 1)/9 \). \( \text{-----(3)} \)
There exist \( A \) and \( B \) such that

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A(p - 1) - Bp_n = (p - 1, p_n). As p_n is a prime there are two possibilities:

(i) \((p - 1, p_n) = 1\) or (ii) \((p - 1, p_n) = p_n\).

In the first case, from (3) we get \(p\) divides \((10 - 1)/9\) or \(p\) divides 1, which is absurd as \(p > 1\). Hence \((p - 1, p_n) = p_n\) or \(p_n\) divides \(p - 1\)

\[p \equiv 1 \pmod{p_n}\]

\[\Rightarrow p^a \equiv 1 \pmod{p_n}\]

on similar lines

\[q^b \equiv 1 \pmod{p_n}\]

hence \(d = p^aq^br^c\ldots \equiv 1 \pmod{p_n}\)

This completes the proof.

COROLLARY: For any prime \(p\) there exists at least one prime \(q\) such that \(q \equiv 1 \pmod{p}\)

Proof: As \(u(n) \equiv 1 \pmod{p_n}\), and also every divisor of \(u(n)\) is \(\equiv 1 \pmod{p_n}\), the corollary stands proved. Also clearly such a ‘\(q\)’ is greater than \(p\), this gives us a proof of the infinitude of the prime numbers as a by product.

REFERENCES:

The Pseudo-Smarandache function was recently defined in a book by Kashihara[1].

**Definition:** For $n > 0$ and an integer, the value of the Pseudo-Smarandache function $Z(n)$ is the smallest number $m$ such that $n$ evenly divides

$$
\sum_{k=1}^{m} k
$$

**Note:** It is well-known that this is equivalent to $n$ evenly dividing $\frac{m(m+1)}{2}$.

The classic functions of number theory are also well-known and have the following definitions.

Let $n$ be any integer greater than 1 with prime factorization

$$
n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}
$$

**Definition:** For $n > 0$ and an integer, the number of divisors function is denoted by $d(n)$. It is well-known that $d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1)$.

**Definition:** For $n > 0$ and an integer, the sum of the positive divisors of $n$ is denoted by $\sigma(n)$. It is well-known that

$$
\sigma(n) = (1 + p_1 + p_1^2 + \ldots + p_1^{\alpha_1}) \ldots (1 + p_k + p_k^2 + \ldots + p_k^{\alpha_k})
$$

**Definition:** For $n > 0$ and an integer, the Euler phi function $\phi(n)$, is the number of integers $k$, $1 \leq k < n$ that are relatively prime to $n$. It is well-known that

$$
\phi(n) = p_1^{\alpha_1-1}(p_1 - 1) \ldots p_k^{\alpha_k-1}(p_k - 1)
$$

Choosing a number $n$ and repeatedly iterating a function is something that has been done many times before. What we will do here now is alternate the iterations between two different functions.
**Example:**
Construct the sequence of function iterations by starting with the number of divisors function and then alternate it with the Pseudo-Smarandache function.

For example, if $n = 3$, then the iterations would be

$$d(3) = 2 \quad Z(2) = 3 \quad d(3) = 2 \quad Z(2) = 3 \ldots$$

Or notationally

$$Z(\ldots d(Z(d(Z(d(n))))) \ldots)$$

For reference purposes, we will refer to this as the Z-d sequence.

Note that for $n = 3$, the sequence of numbers is a two-cycle. This is no accident and it is easy to prove that this is a general result. We do this in a roundabout way.

**Theorem 1:** Let $p$ be a prime, then the Z-d sequence will always be the two cycle

$$2 \ 3 \ 2 \ 3 \ 2 \ 3 \ldots$$

**Proof:** Since $d(p) = 2$ for $p$ any prime and $Z(2) = 3$, which is also a prime, the result is immediate.

This behavior is even more general.

**Theorem 2:** If $n = p_1p_2$, then the Z-d sequence will always be the two cycle

$$4 \ 7 \ 2 \ 3 \ 2 \ 3 \ldots$$

**Proof:** Since $d(n) = 4$ and $Z(4) = 7$ and $d(7) = 2$, which starts the repeated $2 \ 3 \ 2 \ 3 \ldots$ cycle.

**Theorem 3:** If $n = p^2$, the Z-d sequence will always be the two cycle

$$3 \ 2 \ 2 \ 3 \ 2 \ 3 \ 2 \ 3 \ldots$$

**Proof:** Since $d(n) = 3$, $Z(3) = 2$, $d(2) = 2$, $Z(2) = 3$ and $d(3) = 2$. the result follows.

This behavior is a general one that is easy to prove.

**Theorem:** If $n$ is any integer greater than 1, then the Z-d sequence will always reduce to the 2 cycle

$$2 \ 3 \ 2 \ 3 \ 2 \ 3 \ldots$$
Proof: Explicitly testing all numbers \( n \leq 50 \), the result holds. To complete the proof, we rely on two simple lemmas.

**Lemma 1:** If \( n > 50 \), then \( \frac{n}{d(n)} > 3 \).

**Proof:** By a double induction on the number of prime factors and their exponents.

**Basis:** If \( p > 50 \) is prime, then \( d(n) = 2 \).

**Inductive 1:** Assume that \( n = p^k > 50 \) and that \( \frac{n}{d(n)} > 3 \). Then the ratio of
\[
\frac{p^{k+1}}{d(p^{k+1})} = \frac{p^{k+1}}{k+1} > \frac{p^k}{k+1} > 3.
\]

**Inductive 2:** Assume that for \( n = p_1^{a_1} \ldots p_j^{a_j} \), \( \frac{n}{d(n)} > 3 \). Add an additional prime factor to \( n \) to some power. Since the additional prime factor is not necessarily larger than the others, we will call it \( q \) and append it to the end noting that the primes are not necessarily in ascending order.

\[
\frac{n \cdot q^m}{d(n \cdot q^m)} = \frac{n \cdot q^m}{d(n) \cdot d(q^m)}
\]

since \( d \) is multiplicative. Furthermore, it is well-known that \( d(n) \leq n \) for all \( n > 1 \). Therefore, we have
\[
\frac{n \cdot q^m}{d(n \cdot q^m)} \geq \frac{n}{d(n)} > 3.
\]

and the proof is complete.

**Lemma 2:** If \( n > 1 \) is an integer, then the largest value that the ratio \( \frac{Z(n)}{n} \) can have is 2.

**Proof:** It is well-known that if \( n = 2^k \), then \( Z(n) = 2^{k+1} - 1 \). For all other values of \( n \), \( Z(n) \) is at most \( n \).

Therefore, we have an alternating sequence of numbers where one at most doubles the previous value and the other always reduces it by at least a factor of three. Since the one that always reduces by a factor of three is done first this guarantees that the iterations will eventually reduce the value to a number less than 50.

Since the iteration of the \( Z \)-d sequence always goes to the same 2-cycle, the result will be the same if the order of the iterations is reversed to the \( d-Z \) sequence.

Another iterated sequence that can be constructed involves the Euler phi function and the Pseudo-Smarandache function.

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**Definition:** For \( n > 1 \), the Z-phi sequence is the alternating iteration of the Euler phi function followed by the Pseudo-Smarandache function.

\[
Z(\ldots(\phi(Z(\phi(n))))\ldots)
\]

The sequence is the rather boring

\[1 1 1 1 1 \ldots\]

for \( n = 2 \).

This result is not universal, as for all numbers \( 3 \leq n \leq 14 \), the iterations move to the same 2-cycle. However, this is not a universal result, as when \( n = 15 \), the iteration creates the 2-cycle

\[8 15 8 15 8 15 8 15 \ldots\]

which is also the cycle for 16 and 17.

Examining the behavior of the Z-phi iteration for all numbers \( n \leq 254 \), all create either the

\[2 3 2 3 2 3 \ldots\]

or

\[8 15 8 15 8 15 8 \ldots\]

2-cycles. However, when \( n = 255 \), the iteration creates the new 2-cycle

\[128 255 128 255 128 \ldots\]

which is also the cycle for 256.

Creating and running a computer program to check for 2-cycles that are different from the previous three, no new 2-cycle is encountered until for \( n = 65535 \)

\[32768 65535 32768 65535 32768 \ldots\]

which is also the cycle for \( n = 65536 \).
The pattern so far is clear and is summarized in the following chart

<table>
<thead>
<tr>
<th>Pattern</th>
<th>First Instance</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 - 3</td>
<td>$3 = 2^2 - 1$</td>
</tr>
<tr>
<td>8 - 15</td>
<td>$15 = 2^4 - 1$</td>
</tr>
<tr>
<td>128 - 255</td>
<td>$255 = 2^8 - 1$</td>
</tr>
<tr>
<td>32768 - 65535</td>
<td>$65535 = 2^{16} - 1$</td>
</tr>
</tbody>
</table>

which raises several questions.

1) Does the Z-phi sequence always reduce to a 2-cycle of the form $2^{2^k} - 2^{2^k - 1}$ for $k \geq 1$?

2) Does any additional patterns always appear first for $n = 2^{2^k} - 1$?

A computer search was conducted to test these questions.

**Definition:** For $n > 1$, the Z-sigma sequence is the alternating iteration of the sigma, sum of divisors function followed by the Pseudo-Smarandache function.

$$Z(...) \cdot (\sigma(Z(\sigma(n)))) \ldots)$$

For $n = 2$, the Z-sigma sequence creates the 2-cycle

$$3 2 3 2 \ldots$$

and for $3 \leq n \leq 15$, the Z-sigma sequence creates the 2-cycle

$$24 15 24 15 24 15 \ldots$$

However, for $n = 16$, we get our first cycle that is not a 2-cycle, but is in fact a 12-cycle.

$$63 104 64 127 126 312 143 168 48 124 31 32 63 104 64 127 126 312 143 168 48 \ldots$$

The numbers $17 \leq n \leq 19$ all generate the 2-cycle $24 15 24 15 \ldots$, but $n = 20$ generates the 2-cycle

$$42 20 42 20 42 20 \ldots$$

and $n = 21$ generates the 12-cycle

$$63 104 64 127 126 312 143 168 48 124 31 32 63 104 64 127 126 312 143 168 48 \ldots$$
The numbers $20 \leq n \leq 24$ all generate the 2-cycle $24 \ 15$, but $n = 25$ generates the 12-cycle

\[ 63 \ 104 \ 64 \ 127 \ 126 \ 312 \ 143 \ 168 \ 48 \ 124 \ 31 \ 32 \ 63 \ 104 \ 64 \ 127 \ 126 \ 312 \ 143 \ 168 \ 48 \ ... \]

and $n = 26$ the 2-cycle $42 \ 20 \ 42 \ 20 \ ...$

It is necessary to go up to $n = 381$ before we get the new cycle $1023 \ 1536 \ 1023 \ 1536 \ ...$

and a search up to $n = 552,000$ revealed no additional generated cycles. This leads to some obvious additional questions.

1) Is there another cycle generated by the Z-sigma sequence?

2) Is there an infinite number of numbers $n$ that generate the 2-cycle $42 \ 20$?

3) Are there any other numbers $n$ that generate the two cycle $2 \ 3$?

4) Is there a pattern to the first appearance of a new cycle?

In conclusion, the iterated sequences created by alternating a classic function of number theory with the Pseudo-Smarandache functions yields some interesting results that are only touched upon here. The author strongly encourages others to further explore these problems and is interested in hearing of any additional work in this area.

Reference


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THE ALMOST PRESUMABLE MAXIMALITY OF SOME TOPOLOGICAL LEMMA

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Abstract

Some splitting lemma of topological nature provides fundamental information when dealing with dynamics (see [1], pg.79). Because the set involved, namely $X \setminus P_s$, is neither open nor closed, a natural question arise: can this set be modified in order to obtain additional data? Unfortunately, the answer is negative.

For a metric space $X$ which is locally connected and locally compact and for some continuous mapping $f : X \to X$, the set $\omega$-set of each element $x$ of $X$ is given by the formula

$$\omega(x) = \left\{ y \in X | y = \lim_{n \to +\infty} f^{k_n}(x), \lim_{n \to +\infty} k_n = +\infty \right\}.$$  

We also denote by $\omega_j(x)$, $1 \leq j \leq r$, the set

$$\omega_j(x) = \left\{ y \in X | y = \lim_{n \to +\infty} f^{m_n+r^j}(x), \lim_{n \to +\infty} m_n = +\infty \right\}.$$  

Now, $\omega(x)$ can be splitted according to the following lemma.

**Lemma 1**

a) $\omega(x) = \bigcup_{j=1}^{r} \omega_j(x)$;

b) $f(\omega_j(x)) \subseteq \omega_{(j+1) \mod r}$.

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Its proof relies upon the properties of $\omega(x)$.

**Lemma 2** For some nonvoid subset $S$ of $X$ we consider $C$ a component of $X \setminus S$, i.e. a maximal connected set (see [2], pg. 54). Then:

a) $\overline{C}^X \subset C \cup (S \cap \partial^X S)$;

b) $\partial^X C \subset (C \cap \partial^X C) \cup (S \cap \partial^X S)$.

where $\overline{C}^X$ signifies the closure of $C$ under the topology of $X$ while $\partial^X C$ is the boundary of $C$ under the same topology.

**Remark 1** For instance, if $S$ is closed, then $\partial^X C \subset \partial^X S$ as the components of a locally connected space are open.

**Proof.** a) First, let's show that $\overline{C}^X \subset C \cup S$. For $x \in X \setminus (C \cup S) = (X \setminus S) \setminus C$, as $C$ is closed in $X \setminus S$, there will be some open $G \subset X$ such that

$$x \in G \cap (X \setminus S) \subset X \setminus (C \cup S).$$

Obviously,

$$[G \cap (X \setminus S)] \cap C = G \cap C = \emptyset$$

and so

$$x \notin \overline{C}^X.$$  

Further on, let's assume that $x \in \overline{C}^X \cap S$. If $x \in X \setminus \partial^X S$, then $x \notin X \setminus S^X$. There will be some open $W \subset X$ such that

$$x \in W \cap X \setminus S^X = \emptyset.$$  

In particular, $W \cap C = \emptyset$ and so $x \notin \overline{C}^X$.

b) According to a), we have:

$$\overline{C}^X \setminus (X \setminus C^X) = \partial^X C \subset (C \setminus \overline{C}^X) \cup [(S \cap \partial^X S) \setminus \overline{C}^X]$$

$$= (C \setminus \overline{C}^X) \cup (S \cap \partial^X S)$$

because of $S \cap \partial^X S \subset S \subset X \setminus C$.

Obviously,

$$C \setminus \overline{C}^X = (C \setminus \overline{C}^X) \cap \overline{C}^X = C \cap \partial^X C.$$
Remark 2: It's worth noticing that the sets \((C \cap \partial X)\) and \((S \cap \partial X)\) are disjoint; in other words, \(\partial X\) is piecewise-made. Lemma 2 works equally well in any topological space.

Lemma 3 (Melbourne, Delmitz, Golubitsky)

For some nonvoid subset \(S\) of \(X\), we denote by \(\mathcal{P}_s\) the union

\[\mathcal{P}_s(f) = \bigcup_{n=0}^{\infty} (f^n)^{-1}(S)\]

Let \(x\) be some element of \(S\). Then either \(\omega(x) \subset \overline{\mathcal{P}_s}\) or the following are valid:

\(a)\) \(\omega(x) \setminus \mathcal{P}_s\) is covered by finitely many (connected) components \(C_0, \ldots, C_{r-1}\) of \(X \setminus \mathcal{P}_s\);

\(b)\) These components can be ordered so that \(f(C_i) \subset C_{(i+1) \mod r}\);

\(c)\) \(\omega(x) \subset \overline{C_0} \cup \ldots \cup \overline{C_{r-1}}\).

Remark 3: Notice the splitting in relation with lemma 1. As we mentioned in the Abstract, it is quite natural to ask if \(X \setminus \mathcal{P}_s\) can be replaced by the easier-to-work-with \(X \setminus \overline{\mathcal{P}_s}\). The following lemma shows that this would imply no more the presence of finitely many components.

Lemma 4: Let \(S\) be some nonvoid subset of \(X\) which is not dense in \(X\), i.e. \(\overline{S}^X \neq X\). We consider \(C\) a component of \(X \setminus \overline{S}^X\) and \(D\) a component of \(X \setminus S\) such that \(C \subset D\). Then any element \(x\) of \(D \setminus C\) belongs either to \(\partial X S\) or to any other component of \(X \setminus \overline{S}^X\).

Proof. If \(x \notin X \setminus \overline{S}\) then \(x \in (X \setminus S) \cap \overline{S}^X \subset \partial X S\).

An example would be appropriate: in \(R^2\), we denote by \(D(0, r)\) the \(r\)-disk centered in 0. Now, for \(X = \overline{D(0, 2)}^r\), \(S = D(0, 1) \cup (1, 2) \cup (-2, -1)\), we have

\[\overline{S}^R^2 \supset \overline{D(0, 1)}^R^2 \cup (1, 2) \cup (-2, -1)\]

\[D = X \setminus S, \quad C \in \left\{ \left(X \setminus \overline{S}^R^2 \right) \cap (y > 0), \left(X \setminus \overline{S}^R^2 \right) \cap (y < 0) \right\}\]

Further examples can be architected easily even to obtain infinitely many components of \(X \setminus \overline{S}^X\).

In other words, finitely many components of \(X \setminus S\) may include infinitely many components of \(X \setminus \overline{S}^X\).
References


ON THE CONVERGENCE OF THE EULER HARMONIC SERIES

Sabin Tabirca* Tatiana Tabirca*

Abstract: The aim of this article is to study the convergence of the Euler harmonic series. Firstly, the results concerning the convergence of the Smarandache and Erdos harmonic functions are reviewed. Secondly, the Euler harmonic series is proved to be convergent for \( a > 1 \), and divergent otherwise. Finally, the sums of the Euler harmonic series are given.

Key words: series, convergence, Euler function.

The purpose of this article is to introduce the Euler harmonic series and to study its convergence. This problem belongs to a new research direction in Number Theory that is represented by convergence properties of series made with the most used Number Theory functions.

1. Introduction

In this section, the important results concerning the harmonic series for the Smarandache and Erdos function are reviewed.

Definition 1. If \( f : N \rightarrow N^* \) is a function, then the series \( \sum_{n \in I} \frac{1}{f(n)} \) is the harmonic series associated to \( f \) and is shortly named the \( f \) harmonic series.

The convergence of this sort of series has been studied for the Smarandache and Erdos functions so far. Both are important functions in Number Theory being intensely studied. The definitions and main properties of these two important functions are presented in the following:

- The Smarandache function is \( S : N^* \rightarrow N \) defined by

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The Erdos function is \( P : \mathbb{N}^* \rightarrow \mathbb{N} \) defined by

\[
P(n) = \max\{p \in \mathbb{N} \mid n = mp \land p \text{ is prim}\} \quad (\forall n \in \mathbb{N}^* \setminus \{1\}) \quad P(1) = 0.
\]

The main properties of them are:

\[
(\forall a, b \in \mathbb{N}^* ) \quad(a, b) = 1 \Rightarrow S(a \cdot b) = \max\{S(a), S(b)\}, \quad P(a \cdot b) = \max\{P(a), P(b)\}.
\]

\[
(\forall a \in \mathbb{N}^* ) \quad P(a) \leq S(a) \leq a\text{ and the equalities occur iff } a \text{ is prim.}
\]

Erdos [1995] found the relationship between these two functions that is given by

\[
\lim_{n \to \infty} \frac{\# \{ 1 \leq n \mid P(i) < S(i) \} }{n} = 0.
\]

The series \( \sum_{n \geq 1} \frac{1}{S(n)} \) and \( \sum_{n \geq 1} \frac{1}{P(n)} \) are obviously divergent from Equation (4).

The divergence of the series \( \sum_{n \geq 1} \frac{1}{S^2(n)} \) was an open problem for more than ten years. Tabirca [1998] proved the its divergence using an analytical technique. Luca [1999] was able to prove the divergence of the series \( \sum_{n \geq 1} \frac{1}{S^a(n)} \) refining Tabirca's technique. Thus, the Smarandache harmonic series \( \sum_{n \geq 1} \frac{1}{S^a(n)} , a \in \mathbb{R} \) is divergent. Based on this result and on Equation (5), Tabirca [1999] showed that the Erdos harmonic series \( \sum_{n \geq 1} \frac{1}{P^a(n)} , a \in \mathbb{R} \) is divergent too.

Unfortunately, this convergence property has not been studied for the Euler function. This function is defined as follow: \( \varphi : \mathbb{N} \to \mathbb{N} , \quad (\forall n \in \mathbb{N} ) \quad \varphi(n) = \left| \{ k = 1, 2, \ldots, n \mid (k, n) = 1 \} \right| \).

The main properties [Hardy & Wright, 1979] of this function are enumerated in the following:

\[
(\forall a, b \in \mathbb{N} ) \quad (a, b) = 1 \Rightarrow \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b) - \text{the multiplicative property}
\]

\[
a = p_1^{m_1} \cdot p_2^{m_2} \cdot \ldots \cdot p_k^{m_k} \Rightarrow \varphi(a) = a \cdot \left(1 - \frac{1}{p_1} \right) \cdot \left(1 - \frac{1}{p_2} \right) \cdot \ldots \cdot \left(1 - \frac{1}{p_k} \right)
\]

\[
(\forall a \in \mathbb{N} ) \sum_{d \mid a} \varphi(d) = a.
\]

More properties concerning this function can be found in [Hardy & Wright, 1979], [Jones & Jones, 1998] or [Rosen, 1993].

2. The Convergence of the Euler Harmonic Series
In this section, the problem of the convergence for the Euler harmonic series is totally solved. The Euler harmonic series \( \sum_{n=1}^{\infty} \frac{1}{\varphi^a(n)} \), \( a \in \mathbb{R} \) is proved to have the same behavior as the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n^a} \), \( a \in \mathbb{R} \).

**Proposition 1.** The series \( \sum_{n=1}^{\infty} \frac{1}{\varphi^a(n)} \) is divergent for \( a \leq 1 \).

**Proof**

The proof is based on the equation

\[ \varphi(n) \leq n \left( \forall n \geq 1 \right) . \]  

Since \( \frac{1}{\varphi^a(n)} \geq \frac{1}{n^a} \left( \forall n \geq 1 \right) \) and \( \sum_{n=1}^{\infty} \frac{1}{n^a} \) is divergent, it follows that \( \sum_{n=1}^{\infty} \frac{1}{\varphi^a(n)} \) is divergent too.

The convergence of the series for \( a > 1 \) is more difficult than the previous and is studied in the following.

Let us define the function \( d : \mathbb{N}^* \rightarrow \mathbb{N} \) by \( d(n) = \left\lfloor \frac{n}{\text{prime factors of } n} \right\rfloor \). The main properties of this function are given by the next proposition.

**Proposition 2.** The function \( d \) satisfies the following equation:

a) \( d(1) = 0 \).

b) \( \left( \forall a, b \in \mathbb{N}^* \right) (a, b) = 1 \Rightarrow d(a \cdot b) = d(a) + d(b) \).

c) \( \left( \forall n \in \mathbb{N}^* \right) d(n) \leq \log_2(n) \).

**Proof**

Equation (10a.) is obvious.

Let \( a = p_1^m \cdot p_2^m \cdot \ldots \cdot p_s^m \) and \( b = q_1^k \cdot q_2^k \cdot \ldots \cdot q_t^k \) be the prime number decomposition of two relative prime numbers. Thus, \( a \cdot b = p_1^m \cdot p_2^m \cdot \ldots \cdot p_s^m \cdot q_1^k \cdot q_2^k \cdot \ldots \cdot q_t^k \) gives the prime number decomposition for \( ab \). Since the equation \( d(a \cdot b) = s + t \), \( d(a) = s \) and \( d(b) = t \) hold in the above hypotheses, Equation (10b) is true.

Let \( n = p_1^m \cdot p_2^m \cdot \ldots \cdot p_s^m \) be the prime number decomposition of \( n \). Equation (10b) gives the following inequality

\[ d(n) = d(p_1^m \cdot p_2^m \cdot \ldots \cdot p_s^m) = d(p_1^m) + d(p_2^m) + \ldots + d(p_s^m) = 1 + 1 + \ldots + 1 \leq \log_2(p_1^m) + \log_2(p_2^m) + \ldots + \log_2(p_s^m) = \log_2(p_1^m \cdot p_2^m \cdot \ldots \cdot p_s^m) = \log_2(n) \]

that proves Equation (10c).

The following proposition proposes a new inequality concerning the Euler function.
Proposition 3. \((\forall n \geq 1) \varphi(n) \geq \frac{n}{1 + \log_2 n}\).

Proof

Let \(n = p_1^{m_1} \cdot p_2^{m_2} \cdots p_r^{m_r}\) be the prime number decomposition of \(n\) such that \(p_1 < p_2 < \cdots < p_r\). Thus, \(\varphi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right)\) holds. Using the order \(p_1 < p_2 < \cdots < p_r\), it follows that \(2 \leq p_1, 3 \leq p_2, \ldots, d(n) + 1 < p_r\). These inequalities are used as follows:

\[
\varphi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right) \\
\geq n \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{d(n) + 1}\right) = \frac{n}{d(n) + 1}
\]

Equation (10c) used in the last inequality gives \(\varphi(n) \geq \frac{n}{1 + \log_2 n}\).

Proposition 4. If \(a > 1\), then the series \(\sum_{n \geq 1} \left(\frac{1 + \log_2 n}{n}\right)^a\) is convergent.

Proof

The proof uses the following convergence test: "if \((a_n)_{n \geq 0}\) is a decreasing sequence, then the series \(\sum_{n \geq 0} a_n\) and \(\sum_{n \geq 0} a_n^2\) have the same convergence".

Because the sequence \(\left(\frac{1 + \log_2 n}{n}\right)_{n \geq 0}\) is decreasing, the above test can be applied. The condensed series is \(\sum_{n \geq 1} 2^n \cdot \left(\frac{1 + \log_2 2}{2^n}\right)^a = \sum_{n \geq 1} (1 + n)^a \cdot 2^{-a-1} \cdot \sum_{n \geq 1} \frac{n^a}{2^{n(a-1)}}\) that is obviously convergent.

Theorem 5. If \(a > 1\), then the series \(\sum_{n \geq 1} \frac{1}{\varphi^a(n)}\) is convergent.

Proof

According to Proposition 4, the series \(\sum_{n \geq 1} \left(\frac{1 + \log_2 n}{n}\right)^a\) is convergent. Proposition 3 gives the inequality \(\left(\frac{1 + \log_2 n}{n}\right)^a \geq \frac{1}{\varphi^a(n)}\), thus the series \(\sum_{n \geq 1} \frac{1}{\varphi^a(n)}\) is convergent too.
The interesting fact is that the Euler harmonic series has the same behaviour as the classical harmonic series. Therefore, both are convergent for $a > 1$ and both are divergent for $a \leq 1$. The right question is to find information about the sum of the series in the convergence case. Let us denote $E(a) = \sum_{n=1}^{\infty} \frac{1}{\varphi^n(n)}$ the sum of the Euler harmonic series for $a > 1$. These constants can be computed by using a simple computation. They are presented in Table 1 for $a=2,3, \ldots, 7$.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$E(a)$</th>
<th>$a$</th>
<th>$E(a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.39049431</td>
<td>5</td>
<td>2.09837919</td>
</tr>
<tr>
<td>3</td>
<td>2.47619474</td>
<td>6</td>
<td>2.04796102</td>
</tr>
<tr>
<td>4</td>
<td>2.20815078</td>
<td>7</td>
<td>2.02369872</td>
</tr>
</tbody>
</table>

Table 1. The values for $E(a)$.

Unfortunately, none of the above constants are known. Moreover, a relationship between the classical constants ($\pi$, $e$, ...) and them are not obvious. Finding properties concerning the constants $E(a)$ still remains an open research problem.

References

On two notes by M. Bencze

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In vol 10 of this Journal M. Bencze has published two notes on certain inequalities for the Smarandache function. In [2] it is proved that

\[ S\left(\prod_{i=1}^{n} m_i \right) \leq \sum_{i=1}^{n} S(m_i) \]  

(1)

This, in other form is exactly inequality (2) from our paper [5], and follows easily from Le's inequality \( S(ab) \leq S(a) + S(b) \).

In [1] it is proved that

\[ S\left(\prod_{i=1}^{n} a_i b_i \right) \leq \sum_{i=1}^{n} a_i b_i \]  

(2)

The proof follows the method of the problem from [3], i.e.

\[ S(m^n) \leq m \cdot n \]  

(3)

This appears also in [4], [5]. We note here that relation (2) is a direct consequence of (1) and (3), since

\[ S(a_1^{b_1} \cdots a_n^{b_n}) \leq S(a_1^{b_1}) + \ldots + S(a_n^{b_n}) \leq b_1 a_1 + \ldots + b_n a_n \]

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On certain generalizations of the Smarandache function

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1. The famous Smarandache function is defined by $S(n) := \min\{k \in \mathbb{N}^*: n|k!\}, n \geq 1$ positive integer. This arithmetical function is connected to the number of divisors of $n$, and other important number theoretic functions (see e.g. [6], [7], [9], [10]). A very natural generalization is the following one: Let $f : \mathbb{N}^* \to \mathbb{N}^*$ be an arithmetical function which satisfies the following property:

$(P_1)$ For each $n \in \mathbb{N}^*$ there exists at least a $k \in \mathbb{N}^*$ such that $n|f(k)$.

Let $F_f : \mathbb{N}^* \to \mathbb{N}^*$ defined by

$$F_f(n) = \min\{k \in \mathbb{N} : n|f(k)\}. \quad (1)$$

Since every subset of natural numbers is well-ordered, the definition (1) is correct, and clearly $F_f(n) \geq 1$ for all $n \in \mathbb{N}^*$.

Examples. 1) Let $id(k) = k$ for all $k \geq 1$. Then clearly $(P_1)$ is satisfied, and

$$F_{id}(n) = n. \quad (2)$$
2) Let $f(k) = k!$. Then $F_1(n) = S(n)$ - the Smarandache function.

3) Let $f(k) = p_k!$, where $p_k$ denotes the $k$th prime number. Then

$$F_f(n) = \min\{k \in \mathbb{N}^* : n | p_k!\}.$$  \hspace{1cm} (3)

Here $(P_1)$ is satisfied, as we can take for each $n \geq 1$ the least prime greater than $n$.

4) Let $f(k) = \varphi(k)$, Euler’s totient. First we prove that $(P_1)$ is satisfied. Let $n \geq 1$ be given. By Dirichlet’s theorem on arithmetical progressions ([1]) there exists a positive integer $a$ such that $k = an + 1$ is prime (in fact for infinitely many $a$’s). Then clearly $\varphi(k) = an$, which is divisible by $n$.

We shall denote this function by $F_\varphi$.  \hspace{1cm} (4)

5) Let $f(k) = \sigma(k)$, the sum of divisors of $k$. Let $k$ be a prime of the form $an - 1$, where $n \geq 1$ is given. Then clearly $\sigma(n) = an$ divisible by $n$. Thus $(P_1)$ is satisfied. One obtains the arithmetical function $F_{\sigma}$.  \hspace{1cm} (5)

2. Let $A \subset \mathbb{N}^*$, $A \neq \emptyset$ a nonvoid subset of $\mathbb{N}$, having the property:

$(P_2)$ For each $n \geq 1$ there exists $k \in A$ such that $n | k!$.

Then the following arithmetical function may be introduced:

$$S_A(n) = \min\{k \in A : n | k!\}.$$  \hspace{1cm} (6)

Examples. 1) Let $A = \mathbb{N}^*$. Then $S_N(n) \equiv S(n)$ - the Smarandache function.

2) Let $A = \mathbb{N}_1$ = set of odd positive integers. Then clearly $(P_2)$ is satisfied.  \hspace{1cm} (7)

3) Let $A = \mathbb{N}_2$ = set of even positive integers. One obtains a new Smarandache-type function.  \hspace{1cm} (8)

4) Let $A = P = \text{set of prime numbers}$. Then $S_P(n) = \min\{k \in P : n | k!\}$. We shall
denote this function by $P(n)$, as we will consider more closely this function. (9)

3. Let $g : \mathbb{N}^* \rightarrow \mathbb{N}^*$ be a given arithmetical function. Suppose that $g$ satisfies the following assumption:

$$(P_3) \text{ For each } n \geq 1 \text{ there exists } k \geq 1 \text{ such that } g(k) | n.$$ (10)

Let the function $G_g : \mathbb{N}^* \rightarrow \mathbb{N}^*$ be defined as follows:

$$G_g(n) = \max \{ k \in \mathbb{N}^* : g(k) | n \}. \quad (11)$$

This is not a generalization of $S(n)$, but for $g(k) = k!$, in fact one obtains a "dual"-function of $S(n)$, namely

$$G_1(n) = \max \{ k \in \mathbb{N}^* : k! | n \}. \quad (12)$$

Let us denote this function by $S_1(n)$.

There are many other particular cases, but we stop here, and study in more detail some of the above stated functions.

4. The function $P(n)$

This has been defined in (9) by: the least prime $p$ such that $n|p!$. Some values are:

$P(1) = 1$, $P(2) = 2$, $P(3) = 3$, $P(4) = 5$, $P(5) = 5$, $P(6) = 3$, $P(7) = 7$, $P(8) = 5$, $P(9) = 7$, $P(10) = 5$, $P(11) = 11$, ...

**Proposition 1.** For each prime $p$ one has $P(p) = p$, and if $n$ is squarefree, then $P(n) = \text{greatest prime divisor of } n$.

**Proof.** Since $p|p!$ and $p \nmid q!$ with $q < p$, clearly $P(p) = p$. If $n = p_1p_2\ldots p_r$ is squarefree, with $p_1, \ldots, p_r$ distinct primes, if $p_r = \max \{ p_1, \ldots, p_r \}$, then $p_1 \ldots p_r | p_r!$. On the other hand, $p_1 \ldots p_r \nmid q!$ for $q < p_r$, since $p_r \nmid q!$. Thus $p_r$ is the least prime with the required property.
The calculation of $P(p^2)$ is not so simple but we can state the following result:

**Proposition 2.** One has the inequality $P(p^2) \geq 2p + 1$. If $2p + 1 = q$ is prime, then $P(p^2) = q$. More generally, $P(p^m) \geq mp + 1$ for all primes $p$ and all integers $m$. There is equality, if $mp + 1$ is prime.

**Proof.** From $p^2 | (1 \cdot 2 \cdot \ldots \cdot p) (p+1) \ldots (2p)$ we have $p^2 | (2p)!$. Thus $P(p^2) \geq 2p + 1$. One has equality, if $2p + 1$ is prime. By writing $p^m | 1 \cdot 2 \cdot \ldots \cdot p (p+1) \ldots 2p \ldots [(m-1)p + 1] \ldots mp$, where each group of $p$ consecutive terms contains a member divisible by $p$, one obtains $P(p^m) \geq mp + 1$.

**Remark.** If $2p + 1$ is not a prime, then clearly $P(p^2) \geq 2p + 3$.

It is not known if there exist infinitely many primes $p$ such that $2p + 1$ is prime too (see [4]).

**Proposition 3.** The following double inequality is true:

\[
2p + 1 \leq P(p^2) \leq 3p - 1 \tag{13}
\]

\[
mp + 1 \leq P(p^m) \leq (m+1)p - 1 \tag{14}
\]

if $p \geq p_0$.

**Proof.** We use the known fact from the prime number theory ([1], [8]) that for all $a \geq 2$ there exists at least a prime between $2a$ and $3a$. Thus between $2p$ and $3p$ there is at least a prime, implying $P(p^2) \leq 3p - 1$. On the same lines, for sufficiently large $p$, there is a prime between $mp$ and $(m+1)p$. This gives the inequality (14).

**Proposition 4.** For all $n, m \geq 1$ one has:

\[
S(n) \leq P(n) \leq 2S(n) - 1 \tag{15}
\]
and

\[ P(nm) \leq 2[P(n) + P(m)] - 1 \]  \hspace{1cm} (16)

where \( S(n) \) is the Smarandache function.

**Proof.** The left side of (15) is a consequence of definitions of \( S(n) \) and \( P(n) \), while the right-hand side follows from Chebyshev's theorem on the existence of a prime between \( a \) and \( 2a \) (where \( a = S(n) \), when \( 2a \) is not a prime).

For the right side of (16) we use the inequality \( S(mn) \leq S(n) + S(m) \) (see [5]):

\[ P(nm) \leq 2S(nm) - 1 \leq 2[S(n) + S(m)] - 1 \leq 2[P(n) + P(m)] - 1, \]  \hspace{1cm} by (15).

**Corollary.**

\[ \lim_{n \to \infty} \sqrt[n]{P(n)} = 1. \]  \hspace{1cm} (17)

This is an easy consequence of (15) and the fact that \( \lim_{n \to \infty} \sqrt[n]{S(n)} = 1 \). (For other limits, see [6]).

5. The function \( S_*(n) \)

As we have seen in (12), \( S_*(n) \) is in certain sense a dual of \( S(n) \), and clearly

\[ (S_*(n))! |n| (S(n))! \]  \hspace{1cm} which implies

\[ 1 \leq S_*(n) \leq S(n) \leq n \]  \hspace{1cm} (18)

thus, as a consequence,

\[ \lim_{n \to \infty} \sqrt[n]{S_*(n)} = 1. \]  \hspace{1cm} (19)

On the other hand, from known properties of \( S \) it follows that

\[ \liminf_{n \to \infty} \frac{S_*(n)}{S(n)} = 0, \quad \limsup_{n \to \infty} \frac{S_*(n)}{S(n)} = 1. \]  \hspace{1cm} (20)

For odd values \( n \), we clearly have \( S_*(n) = 1 \).
Proposition 5. For \( n \geq 3 \) one has

\[
S_*(n! + 2) = 2
\] (21)

and more generally, if \( p \) is a prime, then for \( n \geq p \) we have

\[
S_*(n! + (p - 1)!) = p - 1.
\] (22)

**Proof.** (21) is true, since \( 2 | (n! + 2) \) and if one assumes that \( k! | (n! + 2) \) with \( k \geq 3 \), then \( 3 | (n! + 2) \), impossible, since for \( n \geq 3 \), \( 3 | n! \). So \( k \leq 2 \), and remains \( k = 2 \).

For the general case, let us remark that if \( n \geq k + 1 \), then, since \( k! | (n! + k!) \), we have

\[
S_*(n! + k!) \geq k.
\]

On the other hand, if for some \( s \geq k + 1 \) we have \( s! | (n! + k!) \), by \( k + 1 \leq n \) we get \( (k + 1)! | (n! + k!) \) yielding \( (k + 1)! | k! \), since \( (k + 1)! | n! \). So, if \( (k + 1)! | k! \) is not true, then we have

\[
S_*(n! + k!) = k.
\] (23)

Particularly, for \( k = p - 1 \) (\( p \) prime) we have \( p \nmid (p - 1)! \).

**Corollary.** For infinitely many \( m \) one has \( S_*(m) = p - 1 \), where \( p \) is a given prime.

Proposition 6. For all \( k, m \geq 1 \) we have

\[
S_*(k!m) \geq k
\] (24)

and for all \( a, b \geq 1 \),

\[
S_*(ab) \geq \max\{S_*(a), S_*(b)\}.
\] (25)

**Proof.** (24) trivially follows from \( k! | (k!m) \), while (25) is a consequence of \( (S_*(a))! | a \Rightarrow (S_*(a))! | (ab) \) so \( S_*(ab) \geq S_*(a) \). This is true if \( a \) is replaced by \( b \), so (25) follows.
Proposition 7. \( S_a[x(x - 1) \ldots (x - a + 1)] \geq a \) for all \( x \geq a \) (\( x \) positive integer). (26)

Proof. This is a consequence of the known fact that the product of \( a \) consecutive integers is divisible by \( a! \).

We now investigate certain properties of \( S_a(a!b!) \). By (24) or (25) we have \( S_a(a!b!) \geq \max\{a, b\} \). If the equation

\[
a!b! = c!
\]

is solvable, then clearly \( S_a(a!b!) = c \). For example, since \( 3! \cdot 5! = 6! \), we have \( S_a(3! \cdot 5!) = 6 \).

The equation (27) has a trivial solution \( c = k!, a = k! - 1, b = k \). Thus \( S_a(k!(k! - 1)!) = k \).

In general, the nontrivial solutions of (27) are not known (see e.g. [3], [1]).

We now prove:

Proposition 8. \( S_a((2k)!(2k + 2)!) = 2k + 2 \), if \( 2k + 3 \) is a prime; \( S_a((2k)!(2k + 2)!) \geq 2k + 4 \), if \( 2k + 3 \) is not a prime. (28) (29)

Proof. If \( 2k + 3 = p \) is a prime, (28) is obvious, since \( (2k + 2)! | (2k)!(2k + 2)! \), but \( (2k + 3)! \nmid (2k)!(2k + 2)! \). We shall prove first that if \( 2k + 3 \) is not prime, then

\[
(2k + 3)|(1 \cdot 2 \ldots (2k))
\]

Indeed, let \( 2k + 3 = ab \), with \( a, b \geq 3 \) odd numbers. If \( a < b \), then \( a < k \), and from \( 2k + 3 \geq 3b \) we have \( b \leq \frac{2}{3}k + 1 < k \). So \( (2k)! \) is divisible by \( ab \), since \( a, b \) are distinct numbers between 1 and \( k \). If \( a = b \), i.e. \( 2k + 3 = a^2 \), then (\(*)\) is equivalent with \( a^2|(1 \cdot 2 \ldots a)(a + 1)\ldots(a^2 - 3) \). We show that there is a positive integer \( k \) such that \( a + 1 < ka \leq a^2 - 3 \) or. Indeed, \( a(a - 3) = a^2 - 3a < a^2 - 3 \) for \( a > 3 \) and \( a(a - 3) > a + 1 \) by \( a^2 > 4a + 1 \), valid for \( a \geq 5 \). For \( a = 3 \) we can verify (\*) directly. Now (\*) gives

\[
(2k + 3)!|(2k)!(2k + 2)!, \text{ if } 2k + 3 \neq \text{prime}
\]

\( \Box \)
implying inequality (29).

For consecutive odd numbers, the product of factorials gives for certain values

\[ S_*(3! \cdot 5!) = 6, \quad S_*(5! \cdot 7!) = 8, \quad S_*(7! \cdot 9!) = 10, \]

\[ S_*(9! \cdot 11!) = 12, \quad S_*(11! \cdot 13!) = 16, \quad S_*(13! \cdot 15!) = 16, \quad S_*(15! \cdot 17!) = 18, \]

\[ S_*(17! \cdot 19!) = 22, \quad S_*(19! \cdot 21!) = 22, \quad S_*(21! \cdot 23!) = 28. \]

The following conjecture arises:

**Conjecture.** \( S_*( (2k - 1)! (2k + 1)! ) = q_k - 1 \), where \( q_k \) is the first prime following \( 2k + 1 \).

**Corollary.** From \( (q_k - 1)! (2k - 1)! (2k + 1)! \) it follows that \( q_k > 2k + 1 \). On the other hand, by \( (2k - 1)! (2k + 1)! (4k)! \), we get \( q_k \leq 4k - 3 \). Thus between \( 2k + 1 \) and \( 4k + 2 \) there is at least a prime \( q_k \). This means that the above conjecture, if true, is stronger than Bertrand’s postulate (Chebyshev’s theorem [1], [8]).

6. Finally, we make some remarks on the functions defined by (4), (5), other functions of this type, and certain other generalizations and analogous functions for further study, related to the Smarandache function.

First, consider the function \( F_\varphi \) of (4), defined by

\[ F_\varphi = \min \{ k \in \mathbb{N}^* : n \mid \varphi(k) \}. \]

First observe that if \( n + 1 = \text{prime} \), then \( n = \varphi(n + 1) \), so \( F_\varphi(n) = n + 1 \). Thus

\[ n + 1 = \text{prime} \Rightarrow F_\varphi(n) = n + 1. \tag{30} \]

This is somewhat converse to the \( \varphi \)-function property

\[ n + 1 = \text{prime} \Rightarrow \varphi(n + 1) = n. \]
Proposition 9. Let \( \phi_n \) be the \( n \)th cyclotomic polynomial. Then for each \( a \geq 2 \) (integer) one has

\[
F_n(n) \leq \phi_n(a) \text{ for all } n.
\] (31)

Proof. The cyclotomic polynomial is the irreducible polynomial of grade \( \varphi(n) \) with integer coefficients with the primitive roots of order \( n \) as zeros. It is known (see [2]) the following property:

\[
n|\varphi(\phi_n(a)) \text{ for all } n \geq 1, \text{ all } a \geq 2.
\] (32)

The definition of \( F_n \) gives immediately inequality (31).

Remark. We note that there exist in the literature a number of congruence properties of the function \( \varphi \). E.g. it is known that \( n|\varphi(a^n - 1) \) for all \( n \geq 1, a \geq 2 \). But this is a consequence of (32), since \( \phi_n(a)|a^n - 1 \), and \( u|v \Rightarrow \varphi(u)|\varphi(v) \) implies (known property of \( \varphi \)) what we have stated.

The most famous congruence property of \( \varphi \) is the following

Conjecture. (D.H. Lehmer (see [4])) If \( \varphi(n)|(n - 1) \), then \( n = \text{prime} \).

Another congruence property of \( \varphi \) is contained in Euler’s theorem: \( m|(a^{\varphi(m)} - 1) \) for \( (a, m) = 1 \). In fact this implies

\[
S(a^{\varphi(m)} - 1) \geq m \text{ for } (a, m!) = 1
\] (33)

and by the same procedure,

\[
S(\varphi(a^n - 1)) \geq n \text{ for all } n.
\] (34)

As a corollary of (34) we can state that

\[
\limsup_{k \to \infty} S(\varphi(k)) = +\infty.
\] (35)
(It is sufficient to take $k = a^n - 1 \to \infty$ as $n \to \infty$).

7. In a completely similar way one can define $F_d(n) = \min\{k : n|d(k)\}$, where $d(k)$ is the number of distinct divisors of $k$. Since $d(2^n - 1) = n$, one has

$$F_d(n) \leq 2^{n-1}. \quad (36)$$

Let now $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ be the canonical factorization of the number $n$. Then Smarandache ([9]) proved that $S(n) = \max\{S(p_1^{\alpha_1}), \ldots, S(p_r^{\alpha_r})\}$.

In the analogous way, we may define the functions

$$S_\varphi(n) = \max\{\varphi(p_1^{\alpha_1}), \ldots, \varphi(p_r^{\alpha_r})\},$$

$$S_\sigma(n) = \max\{\sigma(p_1^{\alpha_1}), \ldots, \sigma(p_r^{\alpha_r})\},$$

etc.

But we can define

$$S'_\varphi(n) = \min\{\varphi(p_1^{\alpha_1}), \ldots, \varphi(p_r^{\alpha_r})\},$$

$$S'(n) = \min\{\varphi(p_1^{\alpha_1}), \ldots, \varphi(p_r^{\alpha_r})\},$$

etc. For an arithmetical function $f$ one can define

$$\Delta_f(n) = \text{l.c.m.}\{f(p_1^{\alpha_1}), \ldots, f(p_r^{\alpha_r})\}$$

and

$$\delta_f(n) = \text{g.c.d.}\{f(p_1^{\alpha_1}), \ldots, f(p_r^{\alpha_r})\}.$$}

For the function $\Delta_\varphi(n)$ the following divisibility property is known (see [8], p.140, Problem 6).

If $(a, n) = 1$, then

$$n|[a^{\Delta_\varphi(n)} - 1]. \quad (37)$$

These functions and many related others may be studied in the near (or further) future.
References


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On the numerical function $S^{-1}_{\text{min}}$

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In [1] one defines $S^{-1}_{\text{min}} : \mathbb{N} \setminus \{1\} \rightarrow \mathbb{N}$, $S^{-1}_{\text{min}}(x) = \min \{S^{-1}(x)\}$, where $S^{-1}(x) = \{a \in \mathbb{N} | S(a) = x\}$, and $S$ is the Smarandache function. For example $S^{-1}(6) = \{2^4, 2^4 \cdot 3, 2^4 \cdot 3^2, 3^2 \cdot 2, 3^2 \cdot 2^2, 3^2 \cdot 3, 2^4 \cdot 3 \cdot 5, 2^3 \cdot 3^2 \cdot 5, 2^3 \cdot 5, 3^2 \cdot 5, 2^4 \cdot 5, 3^2 \cdot 5, 3^2 \cdot 2^4\}$ and $S^{-1}_{\text{min}}(6) = 3^2$.

If $S(x) = n$ one knows that $\text{card} (S^{-1}(n)) = d(n!) - d((n - 1)!)$ where $d$ is the number of divisors of $n$.

If $x$ is a prime number, then $\text{card} (S^{-1}(n)) = d((n - 1)!)$.

We give below a table of the values of $S^{-1}_{\text{min}}(n)$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>12</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^{-1}_{\text{min}}(n)$</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>$3^2$</td>
<td>7</td>
<td>$2^5$</td>
<td>3$^3$</td>
<td>5$^3$</td>
</tr>
<tr>
<td>$n$</td>
<td>16</td>
<td>21</td>
<td>24</td>
<td>27</td>
<td>36</td>
<td>40</td>
<td>52</td>
<td>56</td>
<td>60</td>
</tr>
<tr>
<td>$S^{-1}_{\text{min}}(n)$</td>
<td>$2^{12}$</td>
<td>$7^3$</td>
<td>$3^{10}$</td>
<td>$3^{11}$</td>
<td>$3^{18}$</td>
<td>$5^9$</td>
<td>$13^4$</td>
<td>$7^8$</td>
<td>$5^4$</td>
</tr>
</tbody>
</table>

One knows [2] that if $p < q$ are two prime numbers, and $n > 1$ is a natural number such that $p \cdot q \mid n$, then $p^{l_p(n)} > q^{l_q(n)}$, where $l_p(n)$ is the exponent of $p$ in the prime factors decomposition of $n!$.

According to the above properties we can deduce the calculus formula for function $S^{-1}_{\text{min}}$:

$$S^{-1}_{\text{min}}(n) = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} = p_{p_r}^{l_{p_r}(n) - a_r + 1}$$

(1)

where $p_1 < p_2 < \cdots < p_r$ are the prime numbers in the canonical decomposition of the number $n$.

We list a set of properties of the function $S^{-1}_{\text{min}}$, which result directly from the definition and from formula (1):

1. $S^{-1}_{\text{min}}(p) = p$ if $p$ is a prime number.
2. $S_{\text{min}}^{-1}(p \cdot q) = q^p$ if $p$ and $q$ are prime numbers and $p < q$.

3. $S\left(S_{\text{min}}^{-1}(x)\right) = x$.

4. $S_{\text{min}}^{-1}(q^p) = p \cdot q$ if $p$ and $q$ are prime numbers and $p < q$.

5. $S_{\text{min}}^{-1}(x) < S_{\text{min}}^{-1}(y)$ if $x$ and $y$ contain as the greatest prime factor $p_r$ and $x < y$.

6. The equation $S_{\text{min}}^{-1}(x) = S_{\text{min}}^{-1}(x + 1)$ has no solutions.

7. $S_{\text{min}}^{-1}(S(x))$ is generally not equal to $S(x)$.

8. $\Lambda\left(S_{\text{min}}^{-1}(x)\right) = \log p_r$, where $\Lambda$ is the Mangoldt function.

It is open the problem to find other properties of the function $S_{\text{min}}^{-1}$.

References


On Numbers Where the Values of the Pseudo-Smarandache
Function Of It and The Reversal Are Identical

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The Pseudo-Smarandache function was introduced by Kenichiro Kashihara in a book that is highly recommended[1].

**Definition:** For any $n \geq 1$, the value of the Pseudo-Smarandache function is the smallest integer $m$ such that $n$ evenly divides

$$m = \sum_{k=1}^{m} k.$$  

**Definition:** Let $d = a_1 a_2 \ldots a_k$ be a decimal integer. The **reversal** of $d$, Rev($d$) is the number obtained by reversing the order of the digits of $d$.

Rev($d$) = $a_k a_{k-1} \ldots a_2 a_1$.

If $d$ contains trailing zeros, they are dropped when they become leading zeros.

In this paper, we will look for numbers $n$, such that $Z(n) = Z$(Rev($n$)) and note some of the interesting properties of the solutions. If $n$ is palindromic, then the above property is true by default. Therefore, we will restrict our set of interest to all non-palindromic numbers $n$ such that $Z(n) = Z$(Rev($n$)).

A computer program was written to search for all such $n$ for $1 \leq n \leq 100,000$ and the solutions are summarized below.

$Z(180) = 80 = Z(81)$
$Z(990) = 44 = Z(99)$
$Z(1010) = 100 = Z(101)$
$Z(1089) = 242 = Z(9801)$
$Z(1210) = 120 = Z(121)$
$Z(1313) = 403 = Z(3131)$
$Z(1572) = 392 = Z(2751)$
$Z(1810) = 180 = Z(181)$
$Z(1818) = 404 = Z(8181)$
$Z(2120) = 159 = Z(212)$
Z(2178) = 1088 = Z(8712)
Z(2420) = 120 = Z(242)
Z(2626) = 403 = Z(6262)
Z(2720) = 255 = Z(272)
Z(2997) = 1295 = Z(7992)
Z(3630) = 120 = Z(363)
Z(3636) = 504 = Z(6363)
Z(4240) = 159 = Z(424)
Z(4284) = 1071 = Z(4842)
Z(4545) = 404 = Z(5454)
Z(4640) = 319 = Z(464)
Z(5050) = 100 = Z(505)
Z(6360) = 159 = Z(636)
Z(7170) = 239 = Z(717)
Z(8780) = 439 = Z(878)
Z(9090) = 404 = Z(909)
Z(9490) = 364 = Z(949)
Z(9890) = 344 = Z(989)
Z(13332) = 1616 = Z(23331)
Z(15015) = 714 = Z(51051)
Z(16610) = 604 = Z(1661)
Z(21296) = 6655 = Z(69212)
Z(25520) = 319 = Z(2552)
Z(26664) = 1616 = Z(46662)
Z(27027) = 2079 = Z(72072)
Z(29970) = 1295 = Z(7992)
Z(32230) = 879 = Z(3223)
Z(37730) = 1715 = Z(3773)
Z(39960) = 1295 = Z(6993)
Z(45045) = 2079 = Z(54054)
Z(46662) = 1616 = Z(26664)
Z(49940) = 1815 = Z(4994)
Z(56650) = 824 = Z(5665)
Z(57057) = 2925 = Z(75075)
Z(63630) = 504 = Z(3636)
Z(64460) = 879 = Z(6446)
Z(80080) = 2079 = Z(8008)
Z(80640) = 4095 = Z(4608)
Z(81810) = 404 = Z(1818)
Z(92290) = 3355 = Z(9229)
Z(93390) = 1980 = Z(9339)
Z(96690) = 879 = Z(9669)
Z(97790) = 2540 = Z(9779)
Several items to note from the previous list.

a) Of the 52 solutions discovered, 35 of the numbers have one trailing zero, where many of them are palindromes when the zeros are dropped. While no numbers with two trailing zeros were found, it seems likely that there are such numbers.

**Unsolved Question:** Given that $Z(n) = Z(\text{Rev}(n))$, what is the largest number of trailing zeros that $n$ can have?

The previous question is directly related to the speed with which the Pseudo-Smarandache function grows.

d) Of the 17 remaining numbers, 9 exhibit the pattern $d_1d_2d_1d_2$ or $d_1d_20d_1d_2$.

**Unsolved Question:** Is this a pattern, in the sense that there is an infinite set of numbers $n$, such that $n = d_1d_20\ldots0d_1d_2$ and $Z(n) = Z(\text{Rev}(n))$?

e) Only three of the numbers contain unique nonzero digits and there are none with five digits.

**Unsolved Question:** What is the largest number of unique nonzero digits that a number $n$ can have when $Z(n) = Z(\text{Rev}(n))$?

f) Three of the numbers exhibit the pattern $d_1d_2\ldots d_2d_3$, with the largest interior pattern being three digits in length.

**Unsolved Question:** What is the largest interior pattern of repeating digits $d_2\ldots d_2$ that can appear in a number $n = d_1d_2\ldots d_2d_3$ such that $Z(n) = Z(\text{Rev}(n))$?

Reference

SMARANDACHE RECIPROCAL PARTITION OF UNITY
SETS AND SEQUENCES

(Amarnath Murthy, S.E. (E &T), Well Logging Services, Oil And Natural
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ABSTRACT: Expression of unity as the sum of the reciprocals
of natural numbers is explored. And in this connection
Smarandache Reciprocal partition of unity sets and sequences are
defined. Some results and Inequalities are derived and a few open
problems are proposed.

DISCUSSION:

Define Smarandache Repeatable Reciprocal partition of unity
set as follows:

\[ \text{SRRPS}(n) = \{ x \mid x = (a_1, a_2, \ldots, a_n) \text{ where } \sum_{r=1}^{n} \frac{1}{a_r} = 1. \} \]

\[ f_{RP}(n) = \text{order of the set SRRPS}(n). \]

We have

\[ \text{SRRPS}(1) = \{ (1) \}, f_{RP}(1) = 1. \]

\[ \text{SRRPS}(2) = \{ (2,2) \}, f_{RP}(2) = 1. \]

\[ \text{SRRPS}(3) = \{ (3,3,3), (2,3,6), (2,4,4) \}, f_{RP}(3) = 3, 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} \text{ etc.} \]

\[ \text{SRRPS}(4) = \{ (4,4,4,4), (2,4,6,12), (2,3,7,42), (2,4,5,20), (2,6,6,6), (2,4,8,8), (2,3,12,12), (4,4,3,6), (3,3,6,6), (2,3,10,15) \} \]

\[ f_{RP}(4) = 10. \]

SMARANDACHE REPEATABLE RECIPROCAL PARTITION OF
UNITY SEQUENCE is defined as
Define **SMARANDACHE DISTINCT RECIPROCAL PARTITION OF UNITY SET**

as follows

\[ SDRPS(n) = \{ x \mid x = (a_1, a_2, \ldots, a_n) \text{ where } \sum_{r=1}^{n} (1/a_r) = 1 \text{ and } a_i = a_j \Leftrightarrow i = j \} \]

\[ f_{DP}(n) = \text{order of } SDRPS(n). \]

\[ SDRPS(1) = \{ (1) \} , f_{DP}(1) = 1. \]

\[ SDRPS(2) = \{ \} , f_{RP}(2) = 0. \]

\[ SDRPS(3) = \{ (2,3,6) \} , f_{DP}(3) = 1. \]

\[ SRRPS(4) = \{(2,4,6,12), (2,3,7,42), (2,4,5,20),(2,3,10,15)\} \]

\[ f_{DP}(4) = 4. \]

Smarandache Distinct Reciprocal partition of unity sequences defined as follows

\[ 1 , 0 , 1 , 4 , 12 \ldots \]

the \( n^{th} \) term is \( f_{DP}(n) \).

Following Inequality regarding the function \( f_{DP}(n) \) has been established.

**THEOREM (1.1)**

\[ f_{DP}(n) \geq \sum_{k=3}^{n-1} f_{DP}(k) + (n^2 - 5n + 8)/2 , n > 3 \] \hspace{1cm} \[ \text{(1.1)} \]

This inequality will be established in two steps.
Proposition (1.A)

For every \( n \) there exists a set of \( n \) natural numbers sum of whose reciprocals is 1.

**Proof:** This will be proved by induction. Let the proposition be true for \( n = r \).

Let \( a_1 < a_2 < a_3 < \ldots < a_{n-1} < a_n = k \) be \( r \) distinct natural numbers such that

\[
\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_r} = 1
\]

We have, \( \frac{1}{k} = \frac{1}{(k+1)} + \frac{1}{(k(k+1))} \), which gives us a set of \( r+1 \) distinct numbers \( a_1 < a_2 < a_3 < \ldots < a_{r-1} < k+1 < k(k+1) \), sum of whose reciprocals is 1.

\( P(r) \Rightarrow P(r+1) \), and as \( P(3) \) is true i.e. \( 1/2 + 1/3 + 1/6 = 1 \),

The proposition is true for all \( n \).

This completes the proof of proposition (1.A).

**Note:** If \( a_1, a_2, a_3, \ldots, a_{n-1} \) are \( n-1 \) distinct natural numbers given by

\[
\begin{align*}
a_1 &= 2. \\
a_2 &= a_1 + 1. \\
a_3 &= a_1a_2 + 1 \\
&\vdots \\
a_t &= a_1a_2a_3\ldots a_{t-1} + 1 = a_{t-1}(a_{t-1} - 1) + 1 \\
&\vdots \\
a_{n-2} &= a_1a_2a_3\ldots a_{n-3} + 1 \\
a_{n-1} &= a_1a_2a_3\ldots a_{n-2}
\end{align*}
\]
then these numbers form a set of \((n - 1)\) distinct natural numbers such that

\[
\sum_{t=1}^{n-1} \frac{1}{a_t} = 1.
\]

we have \(a_t = a_{t-1}(a_{t-1} - 1) + 1\) except when \(t = n - 1\) in which case

\[a_{n-1} = a_{n-2}(a_{n-2} - 1)\]

Let the above set be called \textbf{Principle Reciprocal Partition}.

*** It can easily be proved in the above set that

\[a_{2t} \equiv 3 \pmod{10} \quad \text{and} \quad a_{2t+1} \equiv 7 \pmod{10} \quad \text{for} \quad t \geq 1.\]

Consider the principle reciprocal partition for \(n-1\) numbers. Each \(a_t\) contributes one to \(f_{DP}(n)\) if broken into \(a_t + 1\), \(a_t(a_t + 1)\) except for \(t = 1\). (as 2, if broken into 3 and 6, to give \(1/2 = 1/3 + 1/6\), the number 3 is repeated and the condition of all distinct number is not fulfilled). There is a contribution of \(n - 2\) from the principle set to \(f_{DP}(n)\). The remaining \(f_{DP}(n-1) - 1\) members (excluding the principle partition) of \(SDRPS(n-1)\) would contribute at least one each to \(f_{DP}(n)\) (breaking the largest number in each such set into two parts). The contribution to \(f_{DP}(n)\) thus is at least

\[n-2 + f_{DP}(n-1) - 1 = f_{DP}(n-1) + n - 3 \]

\[f_{DP}(n) \geq f_{DP}(n-1) + n - 3 \quad \text{-------}(1.2)\]
Also for each member \((b_1, b_2, \ldots, b_{n-1})\) of SDRPS\((n-1)\) there exists a member of SDRPS\((n)\) i.e. \((2, 2b_1, 2b_2, \ldots, 2b_{n-1})\) as we can see that

\[ 1 = \left(\frac{1}{2}\right)\left(1 + \frac{1}{b_1} + \frac{1}{b_2} + \ldots + \frac{1}{b_{n-1}}\right) = \frac{1}{2} + \frac{1}{2b_1} + \ldots + \frac{1}{2b_{n-1}}. \]

In this way there is a contribution of \(f_{DP}(n-1)\) to \(f_{DP}(n)\). \(-------(1.3)\)

Taking into account all these contributions to \(f_{DP}(n)\) we get

\[ f_{DP}(n) \geq f_{DP}(n-1) + n - 3 + f_{DP}(n-1) \]
\[ f_{DP}(n) \geq 2f_{DP}(n-1) + n - 3 \]
\[ f_{DP}(n) - f_{DP}(n-1) \geq f_{DP}(n-1) + n - 3 \quad \text{-------------(1.4)} \]

from (4) by replacing \(n\) by \(n-1, n-2, \ldots, \) etc. we get

\[ f_{DP}(n-1) - f_{DP}(n-2) \geq f_{DP}(n-2) + n - 4 \]
\[ f_{DP}(n-2) - f_{DP}(n-3) \geq f_{DP}(n-3) + n - 5 \]

\[ \vdots \]
\[ f_{DP}(4) - f_{DP}(3) \geq f_{DP}(3) + 1 \]

summing up all the above inequalities we get

\[ f_{DP}(n) - f_{DP}(3) \geq \sum_{k=3}^{n-1} f_{DP}(k) + \sum_{r=1}^{n-1} r \]
\[ f_{DP}(n) \geq \sum_{k=3}^{n-1} f_{DP}(k) + (n-3)(n-2)/2 + 1 \]
\[ f_{DP}(n) \geq \sum_{k=3}^{n-1} f_{DP}(k) + (n^2 - 5n + 8)/2, \quad n \geq 3 \]

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Remarks: Readers can come up with stronger results as in my opinion the order of $f_{DP}(n)$ should be much more than what has been arrived at. This will be clear from the following theorem.

**THEOREM (1.2):**

If $m$ is a member of an element of SRRPS($n$) say, 

$(a_1, a_2, a_3, \ldots, a_n)$. We have $a_k = m$ for some $k$ and $\sum_{k=1}^{n} \frac{1}{a_k} = 1$.

then $m$ contributes $[\frac{d(m) + 1}{2}]$ elements to SRRPS($n+1$), where the symbol $[\ ]$ stands for integer value and $d(m)$ is the number of divisors of $m$.

**Proof:** For each divisor $d$ of $m$ there corresponds another divisor $m/d = d'$.

**Case-I:** $m$ is not a perfect square. Then $d(m)$ is even and there are $d(m)/2$ pairs of the type $(d, d')$ such that $dd' = m$.

Consider the following identity

$$\frac{1}{p.q} = \frac{1}{p(p+q)} + \frac{1}{q(p+q)} \quad \text{-------------(1.5)}$$

for each divisor pair $(d, d')$ of $m$ we have the following breakup

$$\frac{1}{(d.d')} = \frac{1}{(d(d+d'))} + \frac{1}{(d'(d+d'))}$$

Hence the contribution of $m$ to SRRPS($n+1$) is $d(m)/2$. As $d(m)$ is even $d(m)/2 = [\{d(m) + 1\}/2]$ Also.
Case-II \( m \) is a perfect square. In this case \( d(m) \) is odd and there is a divisor pair \( d=d'= m^{1/2} \). This will contribute one to SRRPS\((n+1)\). The remaining \( \{d(m) - 1\}/2 \) pairs of distinct divisors will contribute as many i.e. say \( \{d(m) - 1\}/2 \). Hence the total contribution in this case would be
\[
\{d(m) - 1\}/2 + 1 = \{d(m) + 1\}/2 = \{d(m) + 1\}/2
\]
Hence \( m \) contributes \( \{d(m) + 1\}/2 \) elements to SRRPS\((n+1)\).

This completes the proof.

Remarks:

1. The total contribution to SRRPS\((n+1)\) by any element of SRRPS\((n)\) is
\[
\sum d(a_k) + 1 \quad \text{(1.6),}
\]
where each \( a_k \) is considered only once irrespective of its' repeated occurrence.

2. In case of SDRPS\((n+1)\), the contribution by an element of SDRPS\((n)\) is given by
\[
\sum_{k=1}^{n} \{d(a_k)/2\} \quad \text{(1.7)}
\]
because the divisor pair \( d=d'= a_k^{1/2} \) does not contribute.

Hence the total contribution of SDRP\((n)\) to generate SDRPS\((n+1)\) is the summation over all the elements of SDRPS\((n)\).

\[
\sum_{f_{DP}(n)}^{n} \sum_{k=1}^{\{d(a_k)/2\}} \quad \text{(1.8)}
\]

Generalizing the above approach.
The readers can further extend this work by considering the following identity

\[ \frac{1}{pqr} = \frac{1}{pq(p+q+r)} + \frac{1}{qr(p+q+r)} + \frac{1}{rp(p+q+r)} \quad -----(1.9) \]

which also suggests

\[ \frac{1}{b_1b_2\ldots b_r} = \sum_{k=1}^{r} \left( \prod_{t=1, t \neq k}^{r} b_t \right) \left( \sum_{s=1}^{r} b_s \right)^{-1} \quad -----(1.10) \]

The above identity can easily be established by just summing up the right hand member.

From (1.10), the contribution of the elements of SDRPS(n) to SDRPS(n+r) can be evaluated if an answer to following tedious queries could be found.

OPEN PROBLEMS:

(1) In how many ways a number can be expressed as the product of 3 of its divisors?

(2) In general in how many ways a number can be expressed as the product of r of its divisors?

(3) Finally in how many ways a number can be expressed as the product of its divisors?
An attempt to get the answers to the above queries leads to the need of the generalization of the theory of partition function.

REFERENCES:


[2] "The Florentine Smarandache" Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.
Partition function $P(n)$ is defined as the number of ways that a positive integer can be expressed as the sum of positive integers. Two partitions are not considered to be different if they differ only in the order of their summands. A number of results concerning the partition function were discovered using analytic functions by Euler, Jacobi, Hardy, Ramanujan and others. Also a number of congruence properties of the function were derived. In the paper Ref.[1]

"SMARANDACHE RECIPROCAL PARTITION OF UNITY SETS AND SEQUENCES"

while dealing with the idea of Smarandache Reciprocal Partitions of unity we are confronted with the problem as to in how many ways a number can be expressed as the product of its divisors. Exploring this lead to the generalization of the theory of partitions.

**DISCUSSION:**

**Definition:** **SMARANDACHE FACTOR PARTITION FUNCTION:**

Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r$ be a set of $r$ natural numbers and $p_1, p_2, p_3, \ldots, p$, be arbitrarily chosen distinct primes then
\( F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) \) called the Smarandache Factor Partition of 
\( (\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) \) is defined as the number of ways in which the number

\[ N = \prod_{i=1}^{r} \alpha_i \]

could be expressed as the product of its' divisors.

Example: \( F(1,2) = ? \),

Let \( p_1 = 2 \) and \( p_2 = 3 \), \( N = p_1 \cdot p_2^2 = 2 \cdot 3^2 = 18 \)

\( N \) can be expressed as the product of its' divisors in following 4 ways:

1. \( N = 18 \)
2. \( N = 9 \times 2 \)
3. \( N = 6 \times 3 \)
4. \( N = 3 \times 3 \times 2 \).

As per our definition \( F(1,2) = 4 \).

It is evident from the definition that \( F(\alpha_1, \alpha_2) = F(\alpha_2, \alpha_1) \) or in general the order of \( \alpha_i \) in \( F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_i \ldots \alpha_r) \) is immaterial. Also the primes \( p_1, p_2, p_3, \ldots p_r \) are dummies and can be chosen arbitrarily.

We start with some elementary results to build up the concept.

**Theorem (2.1):** \( F(\alpha) = P(\alpha) \)

where \( P(\alpha) \) is the number of partitions of \( \alpha \).

**Proof:** Let \( p \) be any prime and \( N = p^\alpha \).

Let \( \alpha = x_1 + x_2 + \ldots + x_m \) be a partition of \( \alpha \).
Then \( N = (p^{x_1})(p^{x_2})(p^{x_3}) \ldots (p^{x_n}) \) is a SFP of \( N \). i.e. each partition of \( \alpha \) contributes one SFP.  \----------(2.1)

Also let one of the SFP of \( N \) be

\[ N = (N_1)(N_2)(N_3) \ldots (N_k) \]

Each \( N_i \) has to be such that \( N_i = p^{a_i} \)

Let \( N_1 = p^{a_1}, N_2 = p^{a_2}, \) etc. \( N_k = p^{a_k} \) then

\[ N = (p^{a_1})(p^{a_2}) \ldots (p^{a_k}) \]

\[ N = p^{(a_1 + a_2 + a_3 + \ldots + a_n)} \]

\[ \Rightarrow \alpha = a_1 + a_2 + \ldots + a_k \]

which gives a partition of \( \alpha \). Obviously each SFP of \( N \) gives one unique partition of \( \alpha \).  \----------(2.2).

From (2.1) and (2.2) we get

\[ F(\alpha) = P(\alpha) \]

**Theorem (2.2)**

\[ F(\alpha, 1) = \sum_{k=0}^{\alpha} P(k) \]

**Proof:** Let \( N = p_1^{\alpha}p_2 \), where \( p_1, p_2 \) are arbitrarily chosen primes.

**Case(1)** Writing \( N = (p_2) p_1^{\alpha} \) keeping \( p_2 \) as a separate entity (one of the factors in the factor partition of \( N \)), would yield \( P(\alpha) \) Smarandache factor partitions (from theorem (2.1)).

**Case(2)** Writing \( N = (p_1p_2). p_1^{\alpha-1} \) keeping \((p_1p_2)\) as a separate entity (one of the factors in the SFP of \( N \)), would yield \( P(\alpha-1) \) SFPs.

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Case (r) In general writing \( N = (p_1^r \cdot p_2) \cdot p_1^{\alpha-r} \) and keeping \((p_1^r \cdot p_2)\) as a separate entity would yield \( P(\alpha-r) \) SFPs.

Contributions towards \( F(N) \) in each case (1), (2), ... (r) are mutually disjoint as \( p_1^r \cdot p_2 \) is unique for a given \( r \), which ranges from zero to \( \alpha \). These are exhaustive also.

Hence

\[
F(\alpha, 1) = \sum_{r=0}^{\alpha} P(\alpha-r)
\]

Let \( \alpha - r = k \)
\[r = 0 \Rightarrow k = \alpha\]
\[r = \alpha \Rightarrow k = 0\]

\[
F(\alpha, 1) = \sum_{k=0}^{\alpha} P(k)
\]

\[
F(\alpha, 1) = \sum_{k=0}^{\alpha} P(k)
\]

This completes the proof of the theorem (2.2)

Some examples:
(1) \( F(3) = P(3) = 3 \), Let \( p = 2 \), \( N = 2^3 = 8 \)

(2) \( F(4, 1) = \sum_{k=0}^{4} P(k) = P(0) + P(1) + P(2) + P(3) + P(4) \)

\[= 1 + 1 + 2 + 3 + 5 = 12\]
Let \( N = 2^4 \times 3 = 48 \) here \( p_1 = 2 \), \( p_2 = 3 \).

The Smarandache factor partitions of 48 are

1. \( N = 48 \)
2. \( N = 24 \times 2 \)
3. \( N = 16 \times 3 \)
4. \( N = 12 \times 4 \)
5. \( N = 12 \times 2 \times 2 \)
6. \( N = 8 \times 6 \)
7. \( N = 8 \times 3 \times 2 \)
8. \( N = 6 \times 4 \times 2 \)
9. \( N = 6 \times 2 \times 2 \times 2 \)
10. \( N = 4 \times 4 \times 3 \)
11. \( N = 4 \times 3 \times 2 \times 2 \)
12. \( N = 3 \times 2 \times 2 \times 2 \times 2 \)

**DEFINITIONS:**

In what follows in the coming pages let us denote (for simplicity)

1. \( F(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = F'(N) \)

where

\[ N = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r \ldots \alpha_n \]

and \( \alpha_r \) is the \( r \)th prime. \( p_1 = 2 \), \( p_2 = 3 \) etc.

2. Also for the case (\( N \) is a square free number)

\[ \alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1 \]

Let us denote

\[ F(1, 1, 1, \ldots) = F(1#n) \]

\[ \leftarrow n \text{- ones} \rightarrow \]

Examples:

\[ F(1#2) = F(1, 1) = F'(6) = 2, \quad 6 = 2 \times 3 = p_1 \times p_2. \]

\[ F(1#3) = F(1, 1, 1) = F'(2 \times 3 \times 5) = F'(30) = 5. \]

3. **Smarandache Star Function**
\[ F'(N) = \sum_{d|N} F'(d_r) \quad \text{where } d_r | N \]

\[ F'(N) = \text{sum of } F'(d_r) \text{ over all the divisors of } N. \]

e.g. \( N = 12 \), divisors are \( 1, 2, 3, 4, 6, 12 \)

\[ F'(12) = F'(1) + F'(2) + F'(3) + F'(4) + F'(6) + F'(12) \]

\[ = 1 + 1 + 1 + 2 + 2 + 4 = 11 \]

\textbf{THEOREM (2.3)}

\[ F'(N) = F'(Np) , \quad (p,N) = 1, \text{ } p \text{ is a prime.} \]

\textbf{PROOF:} We have by definition

\[ F'(N) = \sum_{d|N} F'(d_r) \quad \text{where } d_r | N \]

consider \( d_r \) a divisor of \( N \). \( Np = d_r \cdot (Np/d_r) \)

let \( (Np/d_r) = g(d_r) \), then \( N = d_r \cdot g(d_r) \)

for any divisor \( d_r \) of \( N \), \( g(d_r) \) is unique

\[ d_i = d_j \Leftrightarrow g(d_i) = g(d_j) \]

Considering \( g(d_r) \) as a single term (an entity, not further split into factors) in the SFP of \( N \cdot p \) one gets \( F'(d_r) \) SFPs.

Each \( g(d_r) \) contributes \( F'(d_r) \) factor partitions.

The condition \( p \) does not divide \( N \), takes care that \( g(d_i) \neq d_j \) for any divisor. because \( p \) divides \( g(d_i) \) and \( p \) does not divide \( d_j \).

This ensures that contribution towards \( F'(Np) \) from each \( g(d_r) \) is distinct and there is no repetition. Summing over all \( g(d_r) \) 's we get

\[ F'(Np) = \sum_{d|N} F'(d_r) \]
or
\[ F''(N) = F'(Np) \]
This completes the proof of the theorem (3).

**An application of theorem (2.3)**

Theorem (2.2) follows from theorem (2.3)

To prove
\[ F(\alpha, 1) = \sum_{k=0}^{\alpha} P(k) \]

Let \( N = p^\alpha p_1 \) then \( F(\alpha, 1) = F'(p^\alpha \cdot p_1) \)

from theorem (2.3)

\[ F'(p^\alpha \cdot p_1) = F''(p^\alpha) = \sum_{d/p^\alpha} F'(d_r) \]

The divisors of \( p^\alpha \) are \( p^0, p^1, p^2, \ldots, p^\alpha \)

hence

\[ F'(p^\alpha \cdot p_1) = F'(p^0) + F'(p^1) + \ldots + F'(p^\alpha) \]

\[ = P(0) + P(1) + P(2) + \ldots + P(\alpha -1) + P(\alpha) \]

or

\[ F(\alpha, 1) = \sum_{k=0}^{\alpha} P(k) \]

**THEOREM (2.4):**

\[ F(1# (n+1)) = \sum_{r=0}^{n} \binom{n}{r} F(1#r) \]

**PROOF:** From theorem (2.3) we have \( F'(Np) = F''(N) \), \( p \) does not divide \( N \). Consider the case \( N = p_1p_2p_3\ldots p_n \). We have \( F'(N) \)
Finally we get

\[ F(1#(n+1)) = F'(N) \quad \text{---------(2.3)} \]

The number of divisors of \( N \) of the type \( p_1p_2p_3 \ldots p_r \) (containing exactly \( r \) primes is \( ^nC_r \). Each of the \( ^nC_r \) divisors of the type \( p_1p_2p_3 \ldots p_r \) has the same number of SFPs \( F(1#r) \). Hence

\[ F'(N) = \sum_{r=0}^{n} ^nC_r F(1#r) \quad \text{-----------(2.4)} \]

From (2.3) and (2.4) we get

\[ F(1#(n+1)) = \sum_{r=0}^{n} ^nC_r F(1#r) \]

**NOTE:** It is to be noted that \( F(1#n) \) is the \( n^{th} \) Bell number.

Example: \( F(1#0) = F'(1) = 1 \).

\[ F(1#1) = F'(p_1) = 1. \]

\[ F(1#2) = F'(p_1 p_2) = 2. \]

\[ F(1#2) = F'(p_1 p_2 p_3) = 5. \]

(i) \( p_1 p_2 p_3 \)

(ii) \( (p_1 p_2) \times p_3 \)

(iii) \( (p_1 p_3) \times p_2 \)

(iv) \( (p_2 p_3) \times p_1 \)

(v) \( p_1 \times p_2 \times p_3 \)

Let Theorem (4) be applied to obtain \( F(1#4) \)

\[ F(1#4) = \sum_{r=0}^{3} ^nC_r F(1#r) \]

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\[ r=0 \]

\[
F(1\#4) = 3C_0 F(1\#0) + 3C_1 F(1\#1) + 3C_2 F(1\#2) + 3C_3 F(1\#3)
\]
\[
= 1 \times 1 + 3 \times 1 + 3 \times 2 + 1 \times 5 = 15
\]

\[
F(1\#4) = F'(2 \times 3 \times 5 \times 7) = F'(210) = 15.
\]

(i) 210
(ii) 105 \times 2
(iii) 70 \times 3
(iv) 42 \times 5
(v) 35 \times 6
(vi) 35 \times 3 \times 2
(vii) 30 \times 7
(viii) 21 \times 10
(ix) 21 \times 5 \times 2
(x) 15 \times 14
(xi) 15 \times 7 \times 2
(xii) 14 \times 5 \times 3
(xiii) 10 \times 7 \times 3
(ixiv) 7 \times 6 \times 5
(xv) 7 \times 5 \times 3 \times 2

On similar lines one can obtain

\[
F(1\#5) = 52, F(1\#6) = 203, F(1\#7) = 877, F(1\#8) = 4140.
\]

\[
F(1\#9) = 21,147.
\]

**DEFINITION:**

\[
F^{**}(N) = \sum F^*(d_r) / d_r / N
\]

\[ d_r \] ranges over all the divisors of \( N \).

If \( N \) is a square free number with \( n \) prime factors, let us denote

\[
F^{**}(N) = F^{**}(1\#n)
\]
Example:
\[ F''\*(p_1p_2p_3) = F** (1\#3) = \sum_{d_r/N} F' (d_r) \]

\[ = \binom{3}{0} F''(1) + \binom{3}{1} F'\*(p_1) + \binom{3}{2} F''(p_1p_2) + \binom{3}{3} F''(p_1p_2p_3) \]
\[ F''(1\#3) = 1 + [3F'(1) + F'(p_1)] + 3[F'(1) + 2F'(p_1) + F'(p_1p_2)] \]
\[ + [F'(1) + 3F'(p_1) + 3F'(p_1p_2) + F'(p_1p_2p_3)] \]

\[ F''(1\#3) = 1 + 6 + 15 + 15 = 37. \]

An interesting observation is

(1) \[ F**(1\#0) + F(1\#1) = F(1\#2) \]

or \[ F**(1\#0) + F*(1\#0) = F(1\#2) \]

(2) \[ F**(1\#1) + F(1\#2) = F(1\#3) \]

or \[ F**(1\#1) + F*(1\#1) = F(1\#3) \]

(3) \[ F**(1\#5) + F(1\#6) = F(1\#7) \]

or \[ F**(1\#5) + F*(1\#5) = F(1\#7) \]

which suggests the possibility of

\[ F**(1\#n) + F*(1\#n) = F(1\#(n+2)) \]

A stronger proposition

\[ F'(Np_1p_2) = F''*(N) + F''*(N) \]

is established in theorem (2.5).

**DEFINITION:**

\[ F^{n\#}(N) = \sum_{d_r/N} F'\*(n-1\#) (d_r) \quad n > 1 \]

where \[ F''(N) = \sum_{d_r/N} F'(d_r) \]
and \(d_r\) ranges over all the divisors of \(N\).

**THEOREM (2.5):**

\[
F'(Np_1p_2) = F^*(N) + F^{**}(N)
\]

from theorem (3) we have

\[
F'(Np_1p_2) = F^*(Np_1)
\]

Let \(d_1, d_2, \ldots, d_n\) be all the divisors of \(N\). The divisors of \(Np_1\) would be

\[
d_1, d_2, \ldots, d_n
\]

\[
d_1p_1, d_2p_1, \ldots, d_n p_1
\]

\[
F^*(Np_1) = \left[ F'(d_1) + F'(d_2) + \ldots + F'(d_n) \right] + \left[ F'(d_1 p_1) + F'(d_2 p_1) + \ldots + F'(d_n p_1) \right]
\]

\[
= F^*(N) + \left[ F^*(d_1) + F^*(d_2) + \ldots + F^*(d_n) \right]
\]

\[
F^*(Np_1) = F^*(N) + F^{**}(N) \quad \text{(by definition)}
\]

\[
= F^*(N) + F^2*(N)
\]

This completes the proof of theorem (2.5).

**THEOREM (2.6):**

\[
F'(Np_1p_2p_3) = F^*(N) + 3F^{**}(N) + F^3*(N)
\]

**PROOF:**

From theorem (2.3) we have

\[
F'(Np_1p_2p_3) = F^*(Np_1p_2).
\]

Also if \(d_1, d_2, \ldots, d_n\) be all the divisors of \(N\). Then the
divisors of \( Np_1p_2 \) would be
\[ d_1, \ d_2, \ldots, \ d_n \]
\[ d_1p_1, \ d_2p_1, \ldots, \ d_np_1 \]
\[ d_1p_2, \ d_2p_2, \ldots, \ d_np_2 \]
\[ d_1p_1p_2, \ d_2p_1p_2, \ldots, \ d_np_1p_2 \]

Hence
\[
F'*(Np_1p_2) = [F'(d_1) + F'(d_2) + \ldots + F'(d_n)] +
[F'(d_1p_1) + F'(d_2p_1) + \ldots + F'(d_np_1)] +
[F'(d_1p_2) + F'(d_2p_2) + \ldots + F'(d_np_2)] +
[F'(d_1p_1p_2) + F'(d_2p_1p_2) + \ldots + F'(d_np_1p_2)]
\]
\[
= F'*(N) + 2[F'*(d_1) + F'*(d_2) + \ldots + F'*(d_n)] + S \quad (2.5)
\]

where \( S = [F'(d_1p_1p_2) + F'(d_2p_1p_2) + \ldots + F'(d_np_1p_2) \]

Now from theorem (2.5) we get,
\[
F'(d_1p_1p_2) = F'*(d_1) + F'**(d_1) \quad (1)
\]
\[
F'(d_2p_1p_2) = F'*(d_2) + F'**(d_2) \quad (2)
\]
\[\ldots\]
\[
F'(d_np_1p_2) = F'*(d_n) + F'**(d_n) \quad (n)
\]

on summing up \( (1), (2) \ldots \) upto \( (n) \) we get
\[
S = F'**(N) + F'***(N) \quad (2.6)
\]

substituting the value of \( S \) in (A) and also taking
\[
F'*(d_1) + F'*(d_2) + \ldots + F'*(d_n) = F'**(N)
\]

we get.,
\[ F'(N_{p_1p_2p_3}) = F'^2(N) + 2F'^2(N) + F'^3(N) \]
\[ F'(N_{p_1p_2p_3}) = F'^2(N) + 3F'^2(N) + F'^3(N) \]

This completes the proof of theorem (2.6). The above result which has been observed to follow a beautiful pattern can further be generalized.

REFERENCES:


[3] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.
A GENERAL RESULT ON THE SMARANDACHE STAR FUNCTION

(Amarnath Murthy , S.E. (E & T), Well Logging Services, Oil And Natural Gas Corporation Ltd., Sabarmati, Ahmedabad, India- 380005.)

ABSTRACT: In this paper, the result ( theorem-2.6) Derived in REF. [2], the paper “Generalization Of Partition Function, Introducing ‘Smarandache Factor Partition’ which has been observed to follow a beautiful pattern has been generalized.

DEFINITIONS In [2] we define SMARANDACHE FACTOR PARTITION FUNCTION, as follows:

Let \( \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r \) be a set of \( r \) natural numbers and \( p_1, p_2, p_3, \ldots p_r \) be arbitrarily chosen distinct primes then

\[ F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) \]

called the Smarandache Factor Partition of \( (\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) \) is defined as the number of ways in which the number

\[ N = \frac{\alpha_1}{p_1} \frac{\alpha_2}{p_2} \frac{\alpha_3}{p_3} \ldots \frac{\alpha_r}{p_r} \]

could be expressed as the product of its’ divisors. For simplicity, we denote \( F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) = F'(N) \), where

\[ N = \frac{\alpha_1}{p_1} \frac{\alpha_2}{p_2} \frac{\alpha_3}{p_3} \ldots \frac{\alpha_r}{p_r} \ldots \frac{\alpha_n}{p_n} \]

and \( p_r \) is the \( r \)th prime. \( p_1 = 2, p_2 = 3 \) etc.

Also for the case
we denote
\[ F(1, 1, 1, 1, 1 \ldots) = F(1 \# n) \]
\[ \leftarrow n \text{- ones} \rightarrow \]

Smarandache Star Function

(1) \[ F^{* * } (N) = \sum_{d \mid N} F^{*}(d_r) \]
where \( d_r \mid N \)

(2) \[ F^{* * * } (N) = \sum_{d_r \mid N} F^{* *}(d_r) \]
\( d_r \) ranges over all the divisors of \( N \).

If \( N \) is a square free number with \( n \) prime factors, let us denote
\[ F^{* * * } (N) = F^{* * } (1 \# n) \]

Here we generalise the above idea by the following definition

Smarandache Generalised Star Function

(3) \[ F^{n*}(N) = \sum_{d_r \mid N} F^{(n-1)*}(d_r) \]
\( n > 1 \)

and \( d_r \) ranges over all the divisors of \( N \).

For simplicity we denote
\[ F'(N p_1 p_2 \ldots p_n) = F'(N @ 1 \# n) \]
where \((N, p_i) = 1\) for \( i = 1 \) to \( n \) and each \( p_i \) is a prime.

\( F'(N @ 1 \# n) \) is nothing but the Smarandache factor partition of (a number \( N \) multiplied by \( n \) primes which are coprime to \( N \)).
In [3] a proof of the following result is given:

\[ F'(Np_1 p_2 p_3) = F'*(N) + 3F'*(N) + F'3*(N) \]

The present paper aims at generalising the above result.

**DISCUSSION:**

**THEOREM (3.1)**

\[ F'(N^1@1#n) = F'(Np_1 p_2 \ldots p_n) = \sum_{m=0}^{n} \left[ a_{(n,m)} F^{m*}(N) \right] \]

where

\[ a_{(n,m)} = \left(\frac{1}{m!}\right) \sum_{k=1}^{m} (-1)^{m-k} \cdot m^k C_k \cdot k^n \]

**PROOF:**

Let the divisors of \( N \) be 

\[ d_1, d_2, \ldots, d_k \]

Consider the divisors of \((Np_1 p_2 \ldots p_n)\) arranged as follows

\[ d_1, d_2, \ldots, d_k \quad \text{------say type (0)} \]

\[ d_1p_i, d_2p_i, \ldots, d_kp_i \quad \text{------say type (1)} \]

\[ d_1p_ip_j, d_2p_ip_j, \ldots, d_kp_ip_j \quad \text{------say type (2)} \]

\[ d_1p_ip_j\ldots d_1p_ip_j\ldots, d_kp_ip_j\ldots \quad \text{------say type (t)} \]

(there are \( t \) primes in the term \( d_1p_ip_j\ldots \) apart from \( d_1 \))

\[ d_1p_1p_2\ldots p_n, d_2p_1p_2\ldots p_n, d_n p_1 p_2 \ldots p_n, \quad \text{------say type (n)} \]

There are \( ^nC_0 \) divisors sets of the type (0)

There are \( ^nC_1 \) divisors sets of the type (1)

There are \( ^nC_2 \) divisors sets of the type (2) and so on

There are \( ^nC_t \) divisors sets of the type (t)
There are \(^nC_n\) divisors sets of the type \((n)\)

Let \(Np_1p_2...p_n = M\) then

\[F^*(M) = ^nC_0 [\text{sum of the factor partitions of all the divisors of row (0)}]\]

\[+ ^nC_1 [\text{sum of the factor partitions of all the divisors of row (1)}]\]

\[+ ^nC_2 [\text{sum of the factor partitions of all the divisors of row (2)}]\]

\[+ ...\]

\[+ ^nC_t [\text{sum of the factor partitions of all the divisors of row (t)}]\]

\[+ ...\]

\[+ ^nC_n [\text{sum of the factor partitions of all the divisors of row (n)}]\]

Let us consider the contributions of divisor sets one by one.

Row (0) or type (0) contributes

\[F'(d_1) + F'(d_2) + F'(d_3) + ... + F'(d_n) = F^{**}(N)\]

Row (1) or type (1) contributes

\[\left[F'(d_1p_1) + F'(d_2p_1) + ... + F'(d_kp_1)\right]\]

\[= \left[F^{**}(d_1) + F^{**}(d_2) + ... + F^{**}(d_k)\right]\]

\[= F^{**2}(N)\]

Row (2) or type (2) contributes

\[\left[F'(d_1p_1p_2) + F'(d_2p_1p_2) + ... + F'(d_kp_1p_2)\right]\]

Applying theorem (5) on each of the terms

\[F'(d_1p_1p_2) = F^{**}(d_1) + F^{***}(d_1) \quad ----(1)\]

\[F'(d_2p_1p_2) = F^{**}(d_2) + F^{***}(d_2) \quad ----(2)\]

\[\vdots\]

\[F'(d_kp_1p_2) = F^{**}(d_k) + F^{***}(d_k) \quad ----(k)\]

on summing up \((1), (2) \ldots \text{upto (n)}\) we get

\[F^{**2}(N) + F^{***}(N)\]

At this stage let us denote the coefficients as \(a_{(n,r)}\) etc. say 243
Consider row (t), one divisor set is 

d₀p₁p₂...pₜ , d₀p₁p₂...pₜ , ... dₙ₋₁p₁p₂...pₜ ,

and we have

\[ F'(d₁@1#t) = a_{(t,1)}F'₁(N) + a_{(t,2)}F'₂(N) + ... + a_{(t,t)}F'ₜ(N) \]

summing up both the sides columnwise we get for row (t) or divisors of type (t) one of the \( nCₜ \) divisor sets contributes

\[ a_{(t,1)}F'₂(N) + a_{(t,2)}F'ₜ(N) + ... + a_{(t,t)}F'ₜ(N) \]

similarly for row (n) we get

\[ a_{(n,1)}F'₂(N) + a_{(n,2)}F'ₜ(N) + ... + a_{(n,n)}F'ₜ(N) \]

All the divisor sets of type (0) contribute

\[ nC₀ a_{(0,0)}F'₁(N) \] factor partitions.

All the divisor sets of type (1) contribute

\[ nC₁ a_{(1,1)}F'₂(N) \] factor partitions.

All the divisor sets of type (2) contribute

\[ nC₂ \{a_{(2,1)}F'₂(N) + a_{(2,2)}F'ₜ(N)\} \] factor partitions.

All the divisor sets of type (3) contribute

\[ nC₃\{a_{(3,1)}F'₂(N) + a_{(3,2)}F'ₜ(N) + a_{(3,3)}F'ₜ(N)\} \] factor partitions.
All the divisor sets of row \((t)\) or type \((t)\) contribute

\[ ^nC_t \{ a_{(t,1)}F^{12}(N) + a_{(t,2)}F^{13}(N) + \ldots + a_{(t,t)}F^{(t+1)}(N) \} \]

\[ \ldots \]

All the divisor sets of row \((n)\) or type \((n)\) contribute

\[ ^nC_n \{ a_{(n,1)}F^{12}(N) + a_{(n,2)}F^{13}(N) + \ldots + a_{(n,n)}F^{(n+1)}(N) \} \]

Summing up the contributions from the divisor sets of all the types and considering the coefficient of \(F^m(N)\) for \(m = 1\) to \((n+1)\) we get,

\begin{align*}
\text{coefficient of } F^{12}(N) &= ^nC_1 a_{(1,1)} + ^nC_2 a_{(2,1)} + ^nC_3 a_{(3,1)} + \ldots + ^nC_t a_{(t,1)} + \ldots + ^nC_n a_{(n,1)} \\
&= a_{(n+1,2)} \\
\text{coefficient of } F^{13}(N) &= ^nC_2 a_{(2,2)} + ^nC_3 a_{(3,2)} + ^nC_4 a_{(4,2)} + \ldots + ^nC_t a_{(t,2)} + \ldots + ^nC_n a_{(n,2)} \\
&= a_{(n+1,3)} \\
\text{coefficient of } F^{im}(N) &= a_{(n+1,m)} = ^nC_{m-1} a_{(m-1,m-1)} + ^nC_m a_{(m,m-1)} + \ldots + ^nC_n a_{(n,m-1)} \\
\text{coefficient of } F^{(n+1)}(N) &= a_{(n+1,n+1)} = ^nC_n a_{(n,n)} = ^nC_n \cdot ^{n-1}C_{n-1} a_{(n-1,n-1)} = ^nC_n \cdot ^{n-1}C_{n-1} \ldots \\
2^C_2 a_{(1,1)} &= 1 \\
\text{Consider } a_{(n+1,2)} &= ^nC_1 a_{(1,1)} + ^nC_2 a_{(2,1)} + \ldots + ^nC_t a_{(t,1)} + \ldots + ^nC_n a_{(n,1)}
\[\sum_{i=1}^{n} C_i = 2^n - 1 = (2^{n+1} - 2)/2.\]

Consider \(a_{(n+1,3)}\)

\[= \binom{n}{2} a_{(2,2)} + \binom{n}{3} a_{(3,2)} + \binom{n}{4} a_{(4,2)} + \ldots + \binom{n}{t} a_{(t,2)} + \ldots + \binom{n}{n} a_{(n,2)}\]

\[= \binom{n}{2}(2^1 - 1) + \binom{n}{3}(2^2 - 1) + \binom{n}{4}(2^3 - 1) + \ldots + \binom{n}{n}(2^{n-1} - 1)\]

\[= \binom{n}{2}2^1 + \binom{n}{3}2^2 + \ldots + \binom{n}{n}2^{n-1} - \left\{ \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n} \right\}\]

\[= \frac{1}{2} \left\{ \sum_{r=0}^{n} \binom{n}{r} 2^r - \binom{n}{1} 2^1 - \binom{n}{0} \right\} - \left\{ 2^n - n - 1 \right\}\]

\[= \frac{1}{2} \left\{ 3^n - 2n - 1 \right\} - 2^n + n + 1\]

\[= \frac{1}{2} \left\{ 3^n - 2^n + 1 \right\} \tag{3.1}\]

\[= \frac{1}{13!} \left\{ 3^{n+1} - (3). 2^{n+1} + (3). (1)^{n+1} - (0)^{n+1} \right\}\]

Evaluating \(a_{(n+1,4)}\)

\[a_{(n+1,4)} = \binom{n}{3} a_{(3,3)} + \binom{n}{4} a_{(4,3)} + \ldots + \binom{n}{n} a_{(n,3)}\]

\[= \binom{n}{3}(3^2 + 1 - 2^3)/2 + \binom{n}{4}(3^3 + 1 - 2^4)/2 + \ldots + \binom{n}{n}(3^{n-1} + 1 - 2^n)/2\]

\[= \frac{1}{2} \left\{ 3^2 . \binom{n}{3} + 3^3 . \binom{n}{4} + \ldots + 3^{n-1} . \binom{n}{n} \right\} + \left\{ \binom{n}{3} + \binom{n}{4} + \ldots + \binom{n}{n} \right\}\]

\[= \frac{1}{2} \left\{ \frac{4n - 9n(n-1)/2 - 3n - 1}{2^n} \right\} + \left\{ 2^n - n(n-1)/2 - n - 1 \right\}\]

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\[ a_{n+1,4} = \frac{1}{4!} \left[ (1) \ 4^{n+1} - (4) \ 3^{n+1} + (6) \ 2^{n+1} - (4) \ 1^{n+1} + (0)^{n+1} \right] \]

Observing the pattern we can explore the possibility of

\[ a_{n,r} = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \cdot k^n \quad ------(3.2) \]

which is yet to be established. Now we shall apply induction.

Let the following proposition (3.3) be true for \( r \) and all \( n > r \).

\[ a_{n+1,r} = \frac{1}{r!} \sum_{k=1}^{r} (-1)^{r-k} \cdot C_k \cdot k^{n+1} \quad ------(3.3) \]

Given (3.3) our aim is to prove that

\[ a_{n+1,r+1} = \frac{1}{(r+1)!} \sum_{k=1}^{r+1} (-1)^{r+1-k} \cdot C_k \cdot (k)^{n+1} \]

we have

\[ a_{n+1,r+1} = \binom{n}{r} a_{r,r} + \binom{n}{r+1} a_{r+1,r} + \binom{n}{r+2} a_{r+2,r} + \ldots + \binom{n}{n} a_{n,r} \]

\[ a_{n+1,r+1} = \binom{n}{r} \left( \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \cdot k^r \right) + \binom{n}{r+1} \left( \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \cdot k^{r+1} \right) \]

\[ + \ldots + \binom{n}{n} \left( \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \cdot k^n \right) \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} [(-1)^{r-k} \cdot C_k \left\{ \binom{n}{r} k^r + \binom{n}{r+1} k^{r+1} + \ldots + \binom{n}{n} k^n \right\}] \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} [(-1)^{r-k} \cdot C_k \left\{ \sum_{q=0}^{n} \binom{n}{q} k^q - \sum_{q=0}^{r-1} \binom{n}{q} k^q \right\}] \]

\[ = (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} \cdot C_k (1+k)^n] - (1/r!) \sum_{q=0}^{r-1} [(-1)^{r-k} \cdot C_k \left\{ \sum_{q=0}^{n} \binom{n}{q} k^q \right\}] \]
If we denote the 1\textsuperscript{st} and the second term as $T_1$ and $T_2$, we have

$$a_{(n+1, r+1)} = T_1 - T_2 \quad \text{---------(3.4)}$$

Consider $T_1 = (1/r!) \sum_{k=0}^{r} \left[ (-1)^{r-k} \cdot rC_k \cdot (1+k)^n \right]$

$$= (1/r!) \sum_{k=0}^{r} \left[ (-1)^{r-k} \left\{ \frac{r!}{((k!)(r-k)!)} \right\} (1+k)^n \right]$$

$$= (1/(r+1)!) \sum_{k=0}^{r} \left[ (-1)^{r-k} \cdot \frac{r+1}{((k+1)!(r-k)!)} (1+k)^{n+1} \right]$$

$$= (1/(r+1)!) \sum_{k=0}^{r} \left[ (-1)^{r-k} \cdot rC_{k+1} \cdot (1+k)^{n+1} \right]$$

$$= (1/(r+1)!) \sum_{k=0}^{r} \left[ (-1)^{(r+1)-(k+1)} \cdot rC_{k+1} \cdot (1+k)^{n+1} \right]$$

Let $k+1 = s$, we get $s = 1$ at $k = 0$ and $s = r + 1$ at $k = r$

$$= (1/(r+1)!) \sum_{s=1}^{r+1} \left[ (-1)^{(r+1)-s} \cdot rC_s (s)^{n+1} \right].$$

Replacing $s$ by $k$ we get

$$= (1/(r+1)!) \sum_{k=1}^{r+1} \left[ (-1)^{(r+1)-k} \cdot rC_k (k)^{n+1} \right].$$

In this if we include $k = 0$ case we get

$$T_1 = (1/(r+1)!) \sum_{k=0}^{r+1} \left[ (-1)^{(r+1)-k} \cdot rC_k (k)^{n+1} \right] \quad \text{---------(3.5)}$$

$T_1$ is nothing but the right hand side member of (3.3).

To prove (3.3) we have to prove $a_{(n+1, r+1)} = T_1$

In view of (3.4) our next step is to prove that $T_2 = 0$
\[ T_2 = \frac{1}{r!} \sum_{k=0}^{r} [(-1)^{r-k} \cdot C_k \{ \sum_{q=0}^{r-1} nC_q \cdot k^q \}] \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} [(-1)^{r-k} \cdot C_k \{ nC_0 \cdot k^0 + nC_1 \cdot k^1 + nC_2 \cdot k^2 + \ldots + nC_{r-1} \cdot k^{r-1} \}] \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} [(-1)^{r-k} \cdot C_k ] + nC_1[ \frac{1}{r!} \sum_{k=0}^{r} {(-1)^{r-k} \cdot C_k \cdot k} ] + \]

\[ nC_2[(\frac{1}{r!} \sum_{k=0}^{r} {(-1)^{r-k} \cdot C_k \cdot k^2} ) + \ldots + nC_{r-1}[(\frac{1}{r!} \sum_{k=0}^{r} {(-1)^{r-k} \cdot C_k \cdot k^{r-1}} )] \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} [(-1)^{r-k} \cdot C_k ] + nC_1[ \frac{1}{r!} \sum_{k=0}^{r} {(-1)^{r-k} \cdot C_k \cdot k} ] + \]

\[ [nC_2 \cdot a_{(2,r)} + nC_3 \cdot a_{(3,r)} + \ldots + nC_{r-1} \cdot a_{(r-1,r)} ] \]

\[ = X + Y + Z \quad \text{say where} \]

\[ X = \frac{1}{r!} \sum_{k=0}^{r} [(-1)^{r-k} \cdot C_k ] \quad Y = nC_1[ \frac{1}{r!} \sum_{k=0}^{r} {(-1)^{r-k} \cdot C_k \cdot k} ] \]

\[ Z = [nC_2 \cdot a_{(2,r)} + nC_3 \cdot a_{(3,r)} + \ldots + nC_{r-1} \cdot a_{(r-1,r)} ] \]

We shall prove that \( X = 0 \), \( Y = 0 \), \( Z = 0 \) separately.

(1) \( X = \frac{1}{r!} \sum_{k=0}^{r} [(-1)^{r-k} \cdot C_k ] \)

\[ = \frac{1}{r!} \sum_{k=0}^{r} [(-1)^{r-k} \cdot C_{r-k} ] \]

let \( r - k = w \) then we get at \( k = 0 \) \( w = r \) and at \( k = r \) \( w = 0 \).

\[ = \frac{1}{r!} \sum_{w=r}^{0} [(-1)^w \cdot C_w ] \]

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\[
\begin{align*}
\frac{r}{r!} &= \sum_{w=0}^{r} (-1)^w \cdot \binom{r}{w} \\
&= (1 - 1)^r /r! \\
&= 0
\end{align*}
\]

We have proved that \( X = 0 \)

(2) \[
Y = \binom{n}{1} \left\{ \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot \binom{r}{k} \right\}
\]

\[
= \binom{n}{1} \left\{ \frac{1}{(r-1)!} \sum_{k=1}^{r-1} (-1)^{r-1-(k-1)} \cdot \binom{r-1}{k-1} \right\}
\]

\[
= \binom{n}{1} \left\{ \frac{1}{(r-1)!} (1 - 1)^{r-1} \right\}
\]

\[
= 0
\]

We have proved that \( Y = 0 \)

(3) To prove

\[
Z = [\binom{n}{2} \cdot a_{2,r} + \binom{n}{3} \cdot a_{3,r} + \ldots + \binom{n}{r-1} \cdot a_{(r-1,r)}] = 0 \quad \text{----(3.6)}
\]

Proof:

Refer the matrix

\[
\begin{array}{cccccc}
a_{(1,1)} & a_{(1,2)} & a_{(1,3)} & a_{(1,4)} & \ldots & a_{(1,r)} \\
a_{(2,1)} & a_{(2,2)} & a_{(2,3)} & a_{(2,4)} & \ldots & a_{(2,r)} \\
a_{(3,1)} & a_{(3,2)} & a_{(3,3)} & a_{(3,4)} & \ldots & a_{(3,r)} \\
a_{(4,1)} & a_{(4,2)} & a_{(4,3)} & a_{(4,4)} & \ldots & a_{(4,r)} \\
\end{array}
\]

\[
\begin{array}{cccccc}
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
\]

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The Diagonal elements are underlined. And the the elements above the leading diagonal are shown with bold face.

We have

\[
a_{(1,r)} = \left[ \frac{1}{r!} \sum_{k=0}^{r} \left(-1\right)^{r-k} \binom{r}{k} \right] = \gamma / n C_1 = 0 \text{ for } r > 1
\]

All the elements of the first row except \(a_{(1,1)}\) (the one on the leading diagonal) are zero.

Also

\[
a_{(n+1,r)} = a_{(n,r-1)} + r \cdot a_{(n,r)} \quad \text{--------(3.7)}
\]

(This can be easily established by simplifying the right hand side.)

(7) gives us

\[
a_{(2,r)} = a_{(1,r-1)} + r \cdot a_{(1,r)} = 0 \text{ for } r > 2
\]

i.e. \(a_{(2,r)}\) can be expressed as a linear combination of two elements of the first row (except the one on the leading diagonal)

\[\Rightarrow a_{(2,r)} = 0 \quad r > 2 \]

Similarly \(a_{(3,r)}\) can be expressed as a linear combination of two elements of the second row of the type \(a_{(2,r)}\) with \(r > 3\)

\[\Rightarrow a_{(2,r)} = 0 \quad r > 3 \]

and so on \(a_{(r-1,r)} = 0\)

substituting

\[a_{(2,r)} = a_{(3,r)} = \ldots = a_{(r-1,r)} = 0 \text{ in (6)}\]

we get \(Z = 0\)
With \( X = Y = Z = 0 \) we get \( T_2 = 0 \)
or \( a_{n+1,r+1} = T_1 - T_2 = T_1 \)

from (5) we have

\[
T_1 = (1/(r+1)!) \sum_{k=0}^{r+1} (-1)^{(r+1)} \cdot k \cdot r+1C_k \cdot (k)^{n+1}
\]

which gives

\[
a_{n+1,r+1} = (1/(r+1)!) \sum_{k=0}^{r+1} (-1)^{(r+1)} \cdot k \cdot r+1C_k \cdot (k)^{n+1}
\]

We have proved, if the proposition (3.3) is true for \( r \) it is true for \( r+1 \) as well. We have already verified it for 1, 2, 3 etc. Hence by induction (3.3) is true for all \( n \).

This completes the proof of theorem (3.1).

**Remarks:** This proof is quite lengthy, clumsy and heavy in algebra. The readers can try some analytic, combinatorial approach.

**REFERENCES:**


[3] "The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.

MORE RESULTS AND APPLICATIONS OF THE GENERALIZED SMARANDACHE STAR FUNCTION

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ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION, as follows:

Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r$ be a set of $r$ natural numbers and $p_1, p_2, p_3, \ldots p_r$ be arbitrarily chosen distinct primes then $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ is defined as the number of ways in which the number

$$N = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r \quad p_1 \ p_2 \ p_3 \ldots p_r$$

could be expressed as the product of its' divisors. For simplicity, we denote $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) = F' (N)$, where

$$N = \alpha_1 \alpha_2 \alpha_3 \alpha_r \alpha_n \quad p_1 \ p_2 \ p_3 \ldots p_r \ldots p_n$$

and $p_r$ is the $r^{th}$ prime. $p_1 = 2, p_2 = 3$ etc.

Also for the case

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1$$

Let us denote

$$F ( 1, 1, 1, 1, 1 \ldots ) = F ( 1\#n)$$

$\left\arrow{n - ones} \rightarrow$  

In [2] we define The Generalized Smarandache Star 253
**Function** as follows:

**Smarandache Star Function**

1. \( F'(N) = \sum_{d|N} F'(d_r) \) where \( d_r | N \)

2. \( F''(N) = \sum_{d_r/N} F'(d_r) \)

\( d_r \) ranges over all the divisors of \( N \).

If \( N \) is a square free number with \( n \) prime factors, let us denote

\[ F''(N) = F^{**}(1#n) \]

**Smarandache Generalised Star Function**

3. \( F^{n*}(N) = \sum_{d_r/N} F^{(n-1)*}(d_r) \)

\( n > 1 \) and \( d_r \) ranges over all the divisors of \( N \).

For simplicity we denote

\[ F'(Np_1p_2 \ldots p_n) = F'(N@1#n) \]

where

\( (N,p_i) = 1 \) for \( i = 1 \) to \( n \) and each \( p_i \) is a prime.

\( F'(N@1#n) \) is nothing but the Smarandache factor partition of (a number \( N \) multiplied by \( n \) primes which are coprime to \( N \)).

In [3] I had derived a general result on the Smarandache Generalised Star Function. In the present note some more results and applications of Smarandache Generalised Star Function are explored and derived.

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DISCUSSION:

THEOREM (4.1):

\[ F^{n*}(p^\alpha) = \sum_{k=0}^{\alpha} n^{k-1} C_{n-1} P(\alpha-k) \quad \text{(4.1)} \]

Following proposition shall be applied in the proof of this

\[ \sum_{k=0}^{\alpha} r^{k-1} C_{r-1} = r^r C_r \quad \text{(4.2)} \]

Let the proposition (4.1) be true for \( n = r \) to \( n = 1 \).

\[ F^{r*}(p^\alpha) = \sum_{k=0}^{\alpha} r^{k-1} C_{r-1} P(\alpha-k) \quad \text{(4.3)} \]

\[ F^{(r+1)*}(p^\alpha) = \sum_{t=0}^{\alpha} F^{r*}(p^t) \]

(\( p \) ranges over all the divisors of \( p^\alpha \) for \( t = 0 \) to \( \alpha \))

RHS = \( F^{r*}(p^\alpha) + F^{r*}(p^{\alpha-1}) + F^{r*}(p^{\alpha-2}) + \ldots + F^{r*}(p) + F^{r*}(1) \)

from the proposition (4.3) we have

\[ F^{r*}(p^\alpha) = \sum_{k=0}^{\alpha} r^{k-1} C_{r-1} P(\alpha-k) \]

expanding RHS from \( k = 0 \) to \( \alpha \)

\[ F^{r*}(p^\alpha) = r^{\alpha-1} C_{r-1} P(0) + r^{\alpha-2} C_{r-1} P(1) + \ldots + r^{r} C_{r-1} P(\alpha) \]

similarly

\[ F^{r*}(p^{\alpha-1}) = r^{\alpha-2} C_{r-1} P(0) + r^{\alpha-3} C_{r-1} P(1) + \ldots + r^{r} C_{r-1} P(\alpha-1) \]

\[ F^{r*}(p^{\alpha-2}) = r^{\alpha-3} C_{r-1} P(0) + r^{\alpha-4} C_{r-1} P(1) + \ldots + r^{r} C_{r-1} P(\alpha-2) \]

\[ \ldots \]

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\( F^{r+1}(p) = r^{r-1} C_{r-1} P(0) + r^{r-1} C_{r-1} P(1) \)
\( F^{r+1}(1) = r^{-1} C_{r-1} P(0) \)

Summing up left and right sides separately we find that the

\[ \text{LHS} = F^{(r+1)*}(p^a) \]

The RHS contains \( \alpha + 1 \) terms in which \( P(0) \) occurs, \( \alpha \) terms in which \( P(1) \) occurs etc.

\[ \text{RHS} = \left[ \sum_{k=0}^{\alpha} r^{k-1} C_{r-1} P(0) \right] + \left[ \sum_{k=0}^{\alpha-1} r^{k-1} C_{r-1} P(1) \right] + \ldots + \left[ \sum_{k=0}^{1} r^{k-1} C_{r-1} P(\alpha-1) \right] + \sum_{k=0}^{0} r^{k-1} C_{r-1} P(\alpha) \]

Applying proposition (4.2) to each of the \( \sum \) we get

\[ \text{RHS} = r^{\alpha} C_r P(0) + r^{\alpha-1} C_r P(1) + r^{\alpha-2} C_r P(2) + \ldots + C_r P(\alpha) \]

\[ = \sum_{k=0}^{\alpha} r^{k-1} C_r P(\alpha-k) \]

or

\[ F^{(r+1)*}(p^a) = \sum_{k=0}^{\alpha} r^{k-1} C_r P(\alpha-k) \]

The proposition is true for \( n = r+1 \), as we have

\[ F^{r+1}(p^a) = \sum_{k=0}^{\alpha} P(\alpha-k) = \sum_{k=0}^{\alpha} k C_0 P(\alpha-k) = \sum_{k=0}^{\alpha} k^{1-1} C_{1-1} P(\alpha-k) \]

The proposition is true for \( n = 1 \)

Hence by induction the proposition is true for all \( n \).

This completes the proof of theorem (4.1).

Following theorem shall be applied in the proof of theorem (4.3)

THEOREM (4.2)
\[
\sum_{k=0}^{n-r} \binom{n-r}{k} r^k C_r \ m^k = \binom{n-r}{n} (1+m)^{(n-r)}
\]

**PROOF:**

LHS = \[\sum_{k=0}^{n-r} \binom{n-r}{k} r^k C_r \ m^k\]

= \[\sum_{k=0}^{n-r} \frac{(n)!}{((r+k)!(n-r-k)!)} \cdot \frac{(r+k)!}{((k)!(r)!) \cdot m^k}\]

= \[\sum_{k=0}^{n-r} \frac{(n)!}{((r)!(n-r)!)} \cdot \frac{(n-r)!}{((k)!(n-r-k)!) \cdot m^k}\]

= \[\binom{n-r}{k} \sum_{k=0}^{n-r} \binom{n-r}{k} \ m^k\]

= \[\binom{n-r}{n} (1+m)^{(n-r)}\]

This completes the proof of theorem (4.2)

**THEOREM (4.3):**

\[F_{m*}(1\#n) = \sum_{r=0}^{n} \binom{n}{r} \ m^{n-r} \ F(1\#r)\]

**Proof:**

From theorem (2.4) (ref.[1]) we have

\[F^*(1\#n) = F(1\#(n+1)) = \sum_{r=0}^{n} \binom{n}{r} F(1\#r) = \sum_{r=0}^{n} \binom{n}{r} (1)^{n-r} \ F(1\#r)\]

hence the proposition is true for \(m = 1\).

Let the proposition be true for \(m = s\). Then we have

\[F^s*(1\#n) = \sum_{r=0}^{n} \binom{n}{r} S^{n-r} \ F(1\#r)\]

or

\[F^s*(1\#0) = \sum_{r=0}^{n} \binom{n}{r} S^{0-r} \ F(1\#0)\]

\[F^s*(1\#1) = \sum_{r=0}^{n} \binom{n}{r} S^{1-r} \ F(1\#1)\]

\[F^s*(1\#2) = \sum_{r=0}^{n} \binom{n}{r} S^{2-r} \ F(1\#1)\]

\[F^s*(1\#3) = \sum_{r=0}^{n} \binom{n}{r} S^{3-r} \ F(1\#3)\]

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\[ F^s*(1#0) = 0^0 F(1#0) \quad \text{----(0)} \]
\[ F^s*(1#1) = 1^1 C_0 s^1 F(1#0) + 1^1 C_1 s^0 F(1#1) \quad \text{----(1)} \]
\[ F^s*(1#2) = 2^2 C_0 s^2 F(1#0) + 2^2 C_1 s^1 F(1#1) + 2^2 C_2 s^0 F(1#2) \quad \text{----(2)} \]
\[ \vdots \]
\[ F^s*(1#r) = r^r C_0 s^r F(1#0) + r^r C_1 s^1 F(1#1) + \ldots + r^r C_r s^0 F(1#r) \quad \text{----(r)} \]
\[ \vdots \]
\[ F^s*(1#n) = n^n C_0 s^n F(1#0) + n^n C_1 s^1 F(1#1) + \ldots + n^n C_n s^0 F(1#r) \quad \text{----(n)} \]

Multiplying the \(r^{th}\) equation with \(n^r C_r\) and then summing up we get the RHS as

\[ = \left[ n^n C_0 s^0 + n^n C_1 s^1 + n^n C_2 s^2 + \ldots + n^n C_k s^k + \ldots + n^n C_n s^n \right] F(1#0) \]
\[ \left[ n^n C_1 s^0 + n^n C_2 s^1 + n^n C_3 s^2 + \ldots + n^n C_k s^k + \ldots + n^n C_n s^n \right] F(1#1) \]
\[ \vdots \]
\[ \left[ n^n C_r s^0 + n^n C_{r+1} s^1 + \ldots + n^n C_{r+k} s^k + \ldots + n^n C_n s^n \right] F(1#r) \]
\[ + n^n C_n s^0 \right] F(1#n) \]
\[ = \sum_{r=0}^{n} \sum_{k=0}^{n-r} n^r C_{r+k} s^k \right] F(1#r) \]
\[ = \sum_{r=0}^{n} n^r C_r (1+s)^{n-r} F(1#n) \quad \text{, by theorem (4.2)} \]

LHS = \[ \sum_{r=0}^{n} n^r C_r F^s*(1#r) \]

Let \(N = p_1 p_2 p_3 \ldots p_n\). Then there are \(n^r C_r\) divisors of \(N\) containing exactly \(r\) primes. Then LHS = the sum of the \(s^{th}\) Smarandache star functions of all the divisors of \(N = F^{(s+1)*}(N) = F^{(s+1)*}(1#n)\).

Hence we have

\[ F^{(s+1)*}(1#n) = \sum_{r=0}^{n} n^r C_r (1+s)^{n-r} F(1#n) \]

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\[ F^{s*}(1#0) = ^0C_0 F(1#0) \]  \hspace{1cm} \text{---(0)}

\[ F^{s*}(1#1) = ^1C_0 s^1 F(1#0) + ^1C_1 s^0 F(1#1) \]  \hspace{1cm} \text{---(1)}

\[ F^{s*}(1#2) = ^2C_0 s^2 F(1#0) + ^2C_1 s^1 F(1#1) + ^2C_2 s^0 F(1#2) \]  \hspace{1cm} \text{---(2)}

\[ \vdots \]

\[ F^{s*}(1#r) = ^rC_0 s^r F(1#0) + ^rC_1 s^1 F(1#1) + \ldots + ^rC_r s^0 F(1#r) \]  \hspace{1cm} \text{---(r)}

\[ \vdots \]

\[ F^{s*}(1#n) = ^nC_0 s^n F(1#0) + ^nC_1 s^1 F(1#1) + \ldots + ^nC_n s^0 F(1#n) \]  \hspace{1cm} \text{---(n)}

Multiplying the \( r \)th equation with \( ^nC_r \) and then summing up we get the RHS as

\[ \begin{align*}
&= \left[ ^nC_0 \cdot ^0C_0 s^0 + ^nC_1 \cdot ^1C_0 s^1 + \ldots + ^nC_k \cdot ^kC_0 s^k + \ldots + ^nC_n \cdot ^nC_0 s^n \right] F(1#0) \\
&+ \left[ ^nC_1 \cdot ^1C_1 s^0 + ^nC_2 \cdot ^2C_1 s^1 + \ldots + ^nC_k \cdot ^kC_1 s^k + \ldots + ^nC_n \cdot ^nC_1 s^n \right] F(1#1) \\
&\vdots \\
&\left[ ^nC_r \cdot ^rC_r s^0 + ^nC_{r+1} \cdot ^{r+1}C_r s^1 + \ldots + ^nC_{r+k} \cdot ^{r+k}C_r s^k \ldots + ^nC_n \cdot ^nC_r s^n \right] F(1#r) \\
&+ ^nC_n \cdot ^nC_n s^0 F(1#n)
\end{align*} \]

\[ = \sum_{r=0}^{n} \sum_{k=0}^{n-r} \binom{n}{r+k} \binom{r+k}{r} s^k F(1#r) \]

\[ = \sum_{r=0}^{n} \binom{n}{r} (1+s)^{n-r} F(1#n) \hspace{1cm} \text{by theorem (4.2)} \]

LHS = \[ \sum_{r=0}^{n} \binom{n}{r} F^{s*}(1#r) \]

Let \( N = p_1p_2p_3 \ldots p_n \). Then there are \( ^nC_r \) divisors of \( N \) containing exactly \( r \) primes. Then LHS = the sum of the \( s^{th} \) Smarandache star functions of all the divisors of \( N \).

\[ F^{(s+1)*}(N) = F^{(s+1)*}(1#n) \]

Hence we have

\[ F^{(s+1)*}(1#n) = \sum_{r=0}^{n} \binom{n}{r} (1+s)^{n-r} F(1#n) \]

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which takes the same format
\[ P(s) \Rightarrow P(s+1) \]
and it has been verified that the proposition is true for \( m = 1 \)
hence by induction the proposition is true for all \( m \).
\[ F^m(1#n) = \sum^n_{r=0} m^{n-r} F(1#r) \]
This completes the proof of theorem (4.3)

**NOTE:**
From theorem (3.1) we have
\[ F'(N@1#n) = F'(Np_1p_2 \ldots p_n) = \sum_{m=0}^n a_{n,m} F^m(N) \]
where
\[ a_{n,m} = \frac{1}{m!} \sum_{k=1}^m (-1)^{m-k} \cdot m^k \cdot k^n \]
If \( N = p_1p_2 \ldots p_k \) Then we get
\[ F(1#(k+n) = \sum_{m=0}^n \sum_{t=0}^k \binom{n}{m} \binom{k}{t} m^t F(1#t) \] ------(4.4)
The above result provides us with a formula to express \( B_n \) in terms of smaller Bell numbers. It is in a way generalisation of theorem (2.4) in Ref [5].

**THEOREM(4.4):**
\[ F(\alpha,1#(n+1)) = \sum_{k=0}^\alpha \sum_{r=0}^n n^C_r F(k,1#r) \]

**PROOF:** LHS = \( F(\alpha,1#(n+1)) = F'(p^\alpha p_1p_2p_3 \ldots p_{n+1}) = F^*(p^\alpha p_1p_2p_3 \ldots p_n) + \sum F'( all the divisors containing only p^0) + \sum F'( all the
divisors containing only $p^1) + \sum F' (all the divisors containing only $p^2) + \ldots + \sum F' (all the divisors containing only $p^a)$

$$n \sum_{r=0}^{n} \binom{n}{r} F(0, 1\#r) + \sum_{r=0}^{n} \binom{n}{r} F(1, 1\#r) + \sum_{r=0}^{n} \binom{n}{r} F(2, 1\#r) + \sum_{r=0}^{n} \binom{n}{r} F(3, 1\#r) + \ldots + \sum_{r=0}^{n} \binom{n}{r} F(k, 1\#r) + \ldots + \sum_{r=0}^{n} \binom{n}{r} F(\alpha, 1\#r)$$

$$= \sum_{k=0}^{\alpha} \sum_{r=0}^{n} \binom{n}{r} F(k, 1\#r)$$

This is a reduction formula for $F(\alpha, 1\#(n+1))$

**A Result of significance**

From theorem (3.1) of Ref.: [2], we have

$$F'(p^a@1\#(n+1)) = F(\alpha, 1\#(n+1)) = \sum_{m=0}^{n} a_{(n+1, m)} F^m*(N)$$

where

$$a_{(n+1, m)} = \frac{1}{m!} \sum_{k=1}^{m} (-1)^{m-k} \cdot m C_k \cdot k^{n+1}$$

and

$$F^m*(p^a) = \sum_{k=0}^{\alpha} m+k-1 C_{m-1} P(\alpha-k)$$

This is the first result of some substance, giving a formula for evaluating the number of Smarandache Factor Partitions of numbers representable in a (one of the most simple) particular canonical form. The complexity is evident. The challenging task ahead for the readers is to derive similar expressions for other canonical forms.
REFERENCE


[3] "The Florentine Smarandache" Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.

ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION, as follows: Let \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r \) be a set of \( r \) natural numbers and \( p_1, p_2, p_3, \ldots, p_r \) be arbitrarily chosen distinct primes then \( F(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r) \) called the Smarandache Factor Partition of \( (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r) \) is defined as the number of ways in which the number \( N = \prod_{i=1}^{r} \alpha_i \) could be expressed as the product of its' divisors. For simplicity, we denote \( F(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r) = F(N) \), where

\[
N = \prod_{i=1}^{n} p_i^{\alpha_i}
\]

and \( p_r \) is the \( r \)th prime. \( p_1 = 2, p_2 = 3 \) etc.

Also for the case

\[
\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1
\]

Let us denote

\[
F(1, 1, 1, 1, \ldots) = F(1#n)
\]

\[
\leftarrow n \text{- ones} \rightarrow
\]

In [2] we define The Generalized Smarandache Star Function as follows:
Smarandache Star Function

(1) \( F^*(N) = \sum_{d|r} F^*(d_r) \) where \( d_r | N \)

(2) \( F^{**}(N) = \sum_{d_r|N} F^{**}(d_r) \)

d_r ranges over all the divisors of N.

If N is a square free number with n prime factors, let us denote

\[ F^{**}(N) = F^{**}(1\#n) \]

Smarandache Generalised Star Function

(3) \( F^{n*}(N) = \sum_{d_r|N} F^{(n-1)*}(d_r) \)

\( d_r \) ranges over all the divisors of N.

For simplicity we denote

\[ F'(Np_1p_2\ldots p_n) = F'(N@1\#n) \]

where

\( (N, p_i) = 1 \) for \( i = 1 \) to \( n \) and each \( p_i \) is a prime.

\( F'(N@1\#n) \) is nothing but the Smarandache factor partition of (a number \( N \) multiplied by \( n \) primes which are coprime to \( N \)).

In [2] I had derived a general result on the Smarandache Generalised Star Function. In the present note we define

**SMARANDACHE STAR TRIANGLE'** (SST) and derive some properties of SST.

**DISCUSSION:**
**DEFINITION:** 'SMARANDACHE STAR TRIANGLE' (SST)
As established in [2]

\[ a_{(n,m)} = \frac{1}{m!} \sum_{k=1}^{m} (-1)^{m-k} \cdot m! \cdot \binom{m}{k} \cdot k^n \]  

(1)

we have \( a_{(n,n)} = a_{(n,1)} = 1 \) and \( a_{(n,m)} = 0 \) for \( m > n \). Now if one arranges these elements as follows

\[
\begin{array}{cccc}
  a_{(1,1)}  &   &   &   \\
  a_{(2,1)} & a_{(2,2)} &   &   \\
  a_{(3,1)} & a_{(3,2)} & a_{(3,3)} &   \\
         &       &       &   \\
         &       &       &   \\
  a_{(n,1)} & a_{(n,2)} & \ldots & a_{(n,n-1)} a_{(n,n)} \\
\end{array}
\]

we get the following triangle which we call as the ‘SMARANDACHE STAR TRIANGLE’ in which \( a_{(r,m)} \) is the \( m^{th} \) element of the \( r^{th} \) row and is given by (A) above. It is to be noted here that the elements are the Stirling numbers of the first kind.

\[
\begin{array}{ccccccc}
  1 &   &   &   &   &   &   \\
  1 & 1 &   &   &   &   &   \\
  1 & 3 & 1 &   &   &   &   \\
  1 & 7 & 6 & 1 &   &   &   \\
  1 & 15 & 25 & 10 & 1 &   &   \\
  \ldots &   &   &   &   &   &   \\
\end{array}
\]
Some properties of the SST.

(1) The elements of the first column and the last element of each row is unity.

(2) The elements of the second column are \(2^{n-1} - 1\), where \(n\) is the row number.

(3) Sum of all the elements of the \(n^{th}\) row is the \(n^{th}\) Bell.

**PROOF:**

From theorem (3.1) of Ref. [2] we have

\[
F'(N@1#n) = F'(Np_1p_2\ldots p_n) = \sum_{m=0}^{n} a_{(n,m)} F^{m*}(N)
\]

if \(N = 1\) we get \(F^{m*}(1) = F^{(m-1)*}(1) = F^{(m-2)*}(1) = \ldots = F'(1) = 1\)

hence

\[
F'(p_1p_2\ldots p_n) = \sum_{r=0}^{n} a_{(n,m)}
\]

(4) The elements of a row can be obtained by the following reduction formula

\[
a_{(n+1,m+1)} = a_{(n,m)} + (m+1) \cdot a_{(n+1,m+1)}
\]

instead of having to use the formula (4.5).

(5) If \(N = p\) in theorem (3.1) Ref.[2] we get \(F^{m*}(p) = m + 1\). Hence

\[
F'(pp_1p_2\ldots p_n) = \sum_{m=1}^{n} a_{(n,m)} F^{m*}(N)
\]

or

\[
B_{n+1} = \sum_{m=1}^{n} (m+1) a_{(n,m)}
\]
(6) Elements of second leading diagonal are triangular numbers in their natural order.

(7) If p is a prime, p divides all the elements of the p\textsuperscript{th} row except the 1\textsuperscript{st} and the last, which are unity. This has been established in the following theorem.

**THEOREM (1.1):**

\[ a(p,r) \equiv 0 \pmod{p} \text{ if } p \text{ is a prime and } 1 < r < p \]

**Proof:**

\[ a(p,r) = \frac{1}{r!} \sum_{k=1}^{\infty} (-1)^{r-k} \cdot \binom{r}{k} \cdot k^p \]

Also

\[ a(p,r) = \frac{1}{(r-1)!} \sum_{k=0}^{r-1} (-1)^{r-1-k} \cdot r^{-1} \cdot \binom{r}{k} \cdot (k+1)^{(p-1)} \]

\[ a(p,r) = \frac{1}{(r-1)!} \sum_{k=0}^{r-1} \left[ (-1)^{r-1-k} \cdot r^{-1} \cdot \binom{r}{k} \cdot ((k+1)^{(p-1)} - 1) \right] + \]

\[ (1/(r-1)!) \sum_{k=0}^{r-1} (-1)^{r-1-k} \cdot r^{-1} \cdot \binom{r}{k} \]

applying Fermat's little theorem, we get

\[ a(p,r) = \text{a multiple of } p + 0 \]

\[ \Rightarrow \quad a(p,r) \equiv 0 \pmod{p} \]

**COROLLARY: (1.1)**

\[ F(1\#p) \equiv 2 \pmod{p} \]

\[ a(p,1) = a(p,p) = 1 \]
\[ F(1\#p) = \sum_{k=0}^{p} a_{(p,k)} = \sum_{k=2}^{p-1} a_{(p,k)} + 2 \]

\[ F(1\#p) \equiv 2 \pmod{p} \]

(8) The coefficient of the \(r\)th term \(b_{(n,r)}\) in the expansion of \(x^n\) as

\[ x^n = b_{(n,1)} x + b_{(n,2)} x(x-1) + b_{(n,3)} x(x-1)(x-2) + \ldots + b_{(n,r)} x^r + \ldots + b_{(n,n)} x^n \]

is equal to \(a_{(n,r)}\).

**THEOREM (1.2):** \(B_{3n+2}\) is even else \(B_k\) is odd.

From theorem (2.5) in REF. [1] we have

\[ F'(Nq_1q_2) = F'*(N) + F'^**(N) \]

where \(q_1\) and \(q_2\) are prime.

and \((N,q_1) = (N,q_2) = 1\)

let \(N = p_1p_2p_3\ldots p_n\) then one can write

\[ F'(p_1p_2p_3\ldots p_n) = F'*(p_1p_2p_3\ldots p_n) + F'^**(p_1p_2p_3\ldots p_n) \]

or \(F(1\#(n+2)) = F(1\#(n+1)) + F'^*(1\#n)\)

but \(F'^**(1\#n) = \sum_{r=0}^{n} \binom{n}{r} 2^{n-r} F(1\#r)\)

\[ F'^**(1\#n) = \sum_{r=0}^{n-1} \{ \binom{n}{r} 2^{n-r} F(1\#r) \} + F(1\#n) \]

the first term is an even number say \(= E\), This gives us

\[ F(1\#(n+2)) - F(1\#(n+1)) - F(1\#n) = E, \text{ an even number. } \quad -(1.1) \]

Case- I: \(F(1\#n)\) is even and \(F(1\#(n+1))\) is also even \(\Rightarrow\)

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F(1#(n+2)) is even.

Case -II: F(1#n) is even and F(1#(n+1)) is odd ⇒ F(1#(n+2)) is odd.

again by (1.1) we get

F(1#(n+3)) - F(1#(n+2)) - F(1#(n+1)) = E, ⇒ F(1#(n+3)) is even. Finally we get

F(1#n) is even ⇔ F(1#(n+3)) is even

we know that F(1#2) = 2 ⇒ F(1#2), F(1#5), F(1#8), ... are even

⇒ B_{3n+2} is even else B_k is odd

This completes the proof.

REFERENCES:


[3] "The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.

ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION, as follows:

Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r$ be a set of $r$ natural numbers and $p_1, p_2, p_3, \ldots, p_r$ be arbitrarily chosen distinct primes then $F(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r)$ is defined as the number of ways in which the number

$$N = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_r^{\alpha_r}$$

could be expressed as the product of its' divisors. For simplicity, we denote $F(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r) = F'(N)$, where

$$N = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_r^{\alpha_r} \ldots P_n$$

and $p_r$ is the $r^{th}$ prime. $p_1 = 2, p_2 = 3$ etc.

In the present note we derive a formula for the case $N = p_1^{\alpha} p_2^2$.

DISCUSSION:

Theorem (5.1):

$$F'(p_1^{\alpha} p_2^2) = F(\alpha, 2) = \sum_{k=0}^{\alpha} P(k) + \sum_{j=0}^{\alpha-2} \sum_{i=0}^{j} P(i)$$

where $r = [\alpha/2], \alpha = 2r$ or $\alpha = 2r + 1$
PROOF: Following are the distinct mutually exclusive and exhaustive cases. Only the numbers in the bracket \([ \] \) are to be further decomposed.

Case I: \((p_2) [p_1^\alpha p_2^2] \) gives \( F^*(p_1^\alpha) = \sum_{k=0}^{\alpha} P(i) \)

Case II: \( \{A_1\} \rightarrow (p_2^2) [p_1^\alpha] \rightarrow \rightarrow P(\alpha) \)

\( \{A_2\} \rightarrow (p_2^2 p_1) [p_1^{\alpha-1}] \rightarrow \rightarrow P(\alpha-1) \)

\( \ldots \)

\( \{A_\alpha\} \rightarrow (p_2^2 p_1^\alpha) [p_1^{\alpha-\alpha}] \rightarrow \rightarrow P(\alpha-\alpha) = P(0) \)

Hence Case II contributes \( \sum_{i=0}^{\alpha} P(i) \)

Case III: \( \{B_1\} \rightarrow (p_1 p_2)(p_1 p_2) [p_1^{\alpha-2}] \rightarrow \rightarrow P(\alpha-2) \)

\( \{B_2\} \rightarrow (p_1 p_2)(p_1^2 p_2) [p_1^{\alpha-3}] \rightarrow \rightarrow P(\alpha-3) \)

\( \ldots \)

\( \{B_{\alpha-2}\} \rightarrow (p_1 p_2)(p_1^{\alpha-1} p_2) [p_1^{\alpha-\alpha}] \rightarrow \rightarrow P(\alpha-\alpha) = P(0) \)

Hence Case III contributes \( \sum_{i=0}^{\alpha-2} P(i) \)

Case IV: \( \{C_1\} \rightarrow (p_1^2 p_2)(p_1^2 p_2) [p_1^{\alpha-4}] \rightarrow \rightarrow P(\alpha-4) \)

\( \{C_2\} \rightarrow (p_1^2 p_2)(p_1^3 p_2) [p_1^{\alpha-5}] \rightarrow \rightarrow P(\alpha-5) \)

\( \ldots \)

\( \{C_{\alpha-4}\} \rightarrow (p_1^2 p_2)(p_1^{\alpha-2} p_2) [p_1^{\alpha-\alpha}] \rightarrow \rightarrow P(\alpha-\alpha) = P(0) \)
Hence Case IV contributes \( \sum_{i=0}^{\alpha-4} P(i) \)

\{ NOTE: The factor partition \((p_1^2 p_2) (p_1 p_2) [p_1^{\alpha-3}]\) has already been covered in case III hence is omitted in case IV. The same logic is extended to remaining (following) cases also.\}

**Case V:**  
\{D_1\} \rightarrow (p_1^3 p_2) (p_1^3 p_2) [p_1^{\alpha-6}] \quad \longrightarrow P(\alpha-4)  
\{D_2\} \rightarrow (p_1^3 p_2) (p_1^4 p_2) [p_1^{\alpha-7}] \quad \longrightarrow P(\alpha-5)  
\vdots  
\{D_{\alpha-4}\} \rightarrow (p_1^3 p_2) (p_1^{\alpha-3} p_2) [p_1^{\alpha-\alpha}] \quad \longrightarrow P(\alpha-\alpha) = P(0)

Hence Case V contributes \( \sum_{i=0}^{\alpha-6} P(i) \)

On similar lines case VI contributes \( \sum_{i=0}^{\alpha-8} P(i) \)

we get contributions upto \( \sum_{i=0}^{\alpha-2r} P(i) \)

where \(2r < \alpha < 2r + 1\) or \(r = [\alpha/2]\)

summing up all the cases we get

\[ F'(p_1^\alpha p_2^2) = F(\alpha,2) = \sum_{k=0}^{\alpha} P(k) + \sum_{j=0}^{r} \sum_{i=0}^{\alpha-2j} P(i) \]

where \(r = [\alpha/2]\) \hspace{1cm} \(\alpha = 2r\) or \(\alpha = 2r + 1\)

This completes the proof of theorem (5.1).

**COROLLARY:** (5.1)
\[ F'(p_1^a p_2^2) = \sum_{k=0}^{r} (k+2) [ P(\alpha-2k) + P(\alpha-2k-1)] \quad \text{(5.1)} \]

**Proof:** In theorem (5.1) consider the case \( \alpha = 2r \), we have

\[ F'(p_1^{2r} p_2^2) = F(\alpha, 2) = \sum_{k=0}^{2r} P(k) + \sum_{j=0}^{r} \sum_{i=0}^{\alpha-2j} P(i) \quad \text{(5.2)} \]

Second term on the RHS can be expanded as follows

\[
\begin{align*}
&= P(\alpha) + P(\alpha-1) + P(\alpha-2) + P(\alpha-3) + \ldots + P(2) + P(1) + P(0) \\
&\quad + P(\alpha-2) + P(\alpha-3) + \ldots + P(2) + P(1) + P(0) \\
&\quad \vdots \\
&\quad \vdots \\
&\quad + P(2) + P(1) + P(0) \\
&= \sum_{k=0}^{r} (k+1) [ P(\alpha-2k) + P(\alpha-2k-1)] \\
&= \sum_{k=0}^{r} (k+1) [ P(\alpha-2k) + P(\alpha-2k-1)] \\
&= \sum_{k=0}^{r} (k+1) [ P(\alpha-2k) + P(\alpha-2k-1)]
\end{align*}
\]

\{Here \( P(-1) = 0 \) has been defined.\}

hence

\[
\begin{align*}
F'(p_1^a p_2^2) &= \sum_{k=0}^{r} P(k) + \sum_{k=0}^{r} (k+1) [ P(\alpha-2k) + P(\alpha-2k-1)] \\
&= \sum_{k=0}^{r} (k+1) [ P(\alpha-2k) + P(\alpha-2k-1)]
\end{align*}
\]

Consider the case \( \alpha = 2r+1 \), the second term in the expression (5.2) can be expanded as
\[
P(\alpha) + P(\alpha-1) + P(\alpha-2) + P(\alpha-3) + \ldots + P(2) + P(1) + P(0) \\
= P(\alpha-4) + \ldots + P(2) + P(1) + P(0) \\
\vdots \\
= P(3) + P(2) + P(1) + P(0) \\
\vdots \\
P(1) + P(0)
\]

summing up column wise we get

\[
= [P(\alpha) + P(\alpha-1)] + 2[P(\alpha-2) + P(\alpha-3)] + 3[P(\alpha-4) + P(\alpha-5)] + \ldots \\
+ (r-1)[P(3) + P(2)] + r[P(1) + P(0)].
\]

\[
= \sum_{k=0}^{r} (k+1) [P(\alpha-2k) + P(\alpha-2k-1)], \quad \alpha = 2r+1
\]

on adding the first term, we get

\[
F'(p_1^a p_2^b) = \sum_{k=0}^{r} (k+2) [P(\alpha-2k) + P(\alpha-2k-1)]
\]

(Note here \(P(-1)\) shall not appear.)

Hence for all values of \(\alpha\) we have

\[
F'(p_1^a p_2^b) = \sum_{k=0}^{[\alpha/2]} (k+2) [P(\alpha-2k) + P(\alpha-2k-1)]
\]

This completes the proof of the Corollary (5.1).

REFERENCES:


[2] "The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.

ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION (SFP), as follows:

Let \( \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r \) be a set of \( r \) natural numbers and \( p_1, p_2, p_3, \ldots p_r \) be arbitrarily chosen distinct primes then

\[ F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) \]

called the Smarandache Factor Partition of \( (\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) \) is defined as the number of ways in which the number

\[ N = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r \]

could be expressed as the product of its' divisors. For simplicity, we denote \( F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) = F'(N) \), where

\[ N = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r \ldots \alpha_n \]

and \( p_r \) is the \( r^{th} \) prime. \( p_1 = 2, p_2 = 3 \) etc.

Also for the case

\[ \alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1 \]

we denote

\[ F(1, 1, 1, 1, 1 \ldots) = F(1#n) \]

\[ \leftarrow n \text{- ones} \rightarrow \]

In the present note we define two interesting parameters the
length and extent of an SFP and study the interesting properties they exhibit for square free numbers.

**DISCUSSION:**

**DEFINITION:** Let $F'(N) = R$

**LENGTH:** If we denote each SFP of $N$, say like $F_1, F_2, \ldots, F_R$ arbitrarily and let $F_k$ be the SFP representation of $N$ as the product of its divisors as follows:

$F_k \quad N = (h_1)(h_2)(h_3)(h_4) \ldots (h_t)$, where each $h_i \quad (1 < i < t)$ is an entity in the SFP ‘$F_k$’ of $N$. Then $T(F_k) = t$ is defined as the ‘length’ of the factor partition $F_k$.

e.g. say $60 = 15 \times 2 \times 2$ is a factor partition $F_k$ of 60. Then $T(F_k) = 3$.

$T(F_k)$ can also be defined as one more than the number of product signs in the factor partition.

**EXTENT:** The extent of a number is defined as the sum of the lengths of all the SFPs.

Consider $F(1#3)$

$N = p_1p_2p_3 = 2 \times 3 \times 5 = 30.$

<table>
<thead>
<tr>
<th>SN</th>
<th>Factor Partition</th>
<th>length</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>15 X 2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>10 X 3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>6 X 5</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>5 X 3 X 2</td>
<td>3</td>
</tr>
</tbody>
</table>

Extent (30) = $\sum$ length = 10

We observe that
Consider \( F(1#4) \)

\[ N = 2 \times 3 \times 5 \times 7 = 210 \]

<table>
<thead>
<tr>
<th>SN</th>
<th>Factor Partition</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>210</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>105 \times 2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>70 \times 3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>42 \times 5</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>35 \times 6</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>35 \times 3 \times 2</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>30 \times 7</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>21 \times 10</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>21 \times 5 \times 2</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>15 \times 14</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>15 \times 7 \times 2</td>
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</tr>
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<td>12</td>
<td>14 \times 5 \times 2</td>
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</tr>
<tr>
<td>13</td>
<td>10 \times 7 \times 3</td>
<td>3</td>
</tr>
<tr>
<td>14</td>
<td>7 \times 6 \times 5</td>
<td>3</td>
</tr>
<tr>
<td>15</td>
<td>7 \times 5 \times 3 \times 2</td>
<td>4</td>
</tr>
</tbody>
</table>

\[ \text{Extent}(210) = \sum \text{length} = 37 \]

We observe that

\[ F(1#5) - F(1#4) = 37. = \text{Extent} \{ F(1#4) \} \]

Similarly it has been verified that

\[ F(1#6) - F(1#5) = \text{Extent} \{ F(1#5) \} \]

**CONJECTURE (6.1)**

\[ F(1#(n+1)) - F(1#n) = \text{Extent} \{ F(1#n) \} \]

**CONJECTURE (6.2)**

\[ F(1#(n+1)) = \sum_{r=0}^{n} \text{Extent} \{ F(1#r) \} \]

Motivation for this conjecture:
If conjecture (1) is true then we would have

\[ F(1#2) - F(1#1) = \text{Extent} \{ F(1#1) \} \]

\[ F(1#3) - F(1#2) = \text{Extent} \{ F(1#2) \} \]

\[ F(1#4) - F(1#3) = \text{Extent} \{ F(1#3) \} \]

\[ \vdots \]

\[ F(1#(n+1)) - F(1#n) = \text{Extent} \{ F(1#n) \} \]

Summing up we would get

\[ n \]

\[ F(1#(n+1)) - F(1#1) = \sum_{r=1}^{n} \text{Extent} \{ F(1#r) \} \]

\[ F(1#1) = 1 = \text{Extent} \{ F(1#0) \} \text{ can be taken, hence we get} \]

\[ F(1#(n+1)) = \sum_{r=0}^{n} \text{Extent} \{ F(1#r) \} \]

**Another Interesting Observation:**

Given below is the chart of \( r \) versus \( w \) where \( w \) is the number of SFPs having same length \( r \).

\[
\begin{array}{c|c}
        & 1 \\
\hline
r & 1 \\
\hline
w & 1 \\
\end{array}
\]

\[
\begin{array}{c|c}
        & 1 \\
\hline
r & 1 \\
\hline
w & 1 \\
\end{array}
\]

\[
\begin{array}{c|cc}
        & 2 & 2 \\
\hline
r & 1 & 1 \\
\hline
w & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|ccc}
        & 2 & 3 & 3 \\
\hline
r & 1 & 1 & 1 \\
\hline
w & 1 & 1 & 1 \\
\end{array}
\]

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\[ F(1#4) = 15, \sum r \cdot w = 37 \]

<table>
<thead>
<tr>
<th>r</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>w</td>
<td>1</td>
<td>7</td>
<td>6</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

\[ F(1#5) = 52, \quad \sum r \cdot w = 151 \]

<table>
<thead>
<tr>
<th>r</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>w</td>
<td>1</td>
<td>15</td>
<td>25</td>
<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

The interesting observation is the row of \( w \) is the same as the \( n^{th} \) row of the **SMARANDACHE STAR TRIANGLE**. (ref.: [4])

**CONJECTURE (6.3)**

\[ w_r = a_{(n,r)} = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \cdot k^n \]

where \( w_r \) is the number of SFPs of \( F(1#n) \) having length \( r \).

**Further Scope:** One can study the length and contents of other cases (other than the square-free numbers.) explore for patterns if any.

**REFERENCES:**

4. "The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA."
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$$N = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r$$

$P_1 P_2 P_3 \ldots P_r$ could be expressed as the product of its' divisors. For simplicity, we denote $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ = $F(N)$, where

$$N = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r \alpha_n$$

$P_1 P_2 P_3 \ldots P_r \ldots P_n$ and $p_r$ is the $r^{th}$ prime. $p_1 = 2$, $p_2 = 3$ etc.

In this note another result pertaining to SFPs has been derived.

DISCUSSION:

Let

$$N = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r$$

$P_1 P_2 P_3 \ldots P_r$

(1) $L(N)$ = length of that factor partition of $N$ which contains the maximum number of terms. In this case we have
\[ L(N) = \sum_{i=1}^{r} \alpha_i \]  

(2)  
\[ A_{L(N)} = \text{A set of } L(N) \text{ distinct primes.} \]

(3)  
\[ B(N) = \{ p: p \mid N, p \text{ is a prime. } \} \]
\[ B(N) = \{ p_1, p_2, \ldots, p_r \} \]

(4)  
\[ \Psi[N, A_{L(N)}] = \{ x \mid d(x) = N \text{ and } B(x) \subseteq A_{L(N)} \} , \text{ where } d(x) \text{ is the number of divisors of } x. \]

To derive an expression for the order of the set \( \Psi[N, A_{L(N)}] \) defined above.

There are \( F'(N) \) factor partitions of \( N \). Let \( F_1 \) be one of them.

\[ F_1 \longrightarrow N = s_1Xs_2Xs_3X\ldotsXs_t. \]

if

\[ \theta = \begin{pmatrix} s_1^{-1} & s_2^{-1} & s_3^{-1} & s_t^{-1} & 0 & 0 & 0 \\ p_1 & p_2 & p_3 & \ldots & p_t & p_{t+1} & p_{t+2} \ldots & p_{L(N)} \end{pmatrix} \]

where \( p_t \in A_{L(N)} \), then \( \theta \in \Psi[N, A_{L(N)}] \) for

\[ d(\theta) = s_1Xs_2Xs_3X\ldotsXs_tX1X1X1\ldots = N \]

Thus each factor partition of \( N \) generates a few elements of \( \Psi \).

Let \( E(F_1) \) denote the number of elements generated by \( F_1 \).

\[ F_1 \longrightarrow N = s_1Xs_2Xs_3X\ldotsXs_t. \]

multiplying the right member with unity as many times as required to make the number of terms in the product equal to \( L(N) \).

\[ N = \prod_{k=1}^{L(N)} s_k \]  

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where \( s_{t+1} = s_{t+2} = s_{t+3} = \ldots = s_{L(N)} = 1 \)

Let \( x_1 \) s's are equal
\( x_2 \) s's are equal
\( \ldots \)
\( x_m \) s's are equal

such that \( x_1 + x_2 + x_3 + \ldots + x_m = L(N) \). Where any \( x_i \) can be unity also.

Then we get

\[
E(F_k) = \frac{L(N)!}{(x_1)! (x_2)! (x_3)! \ldots (x_m)!}
\]

summing over all the factor partitions we get

\[
O(Y[N, A_{L(N)}]) = \sum_{k=1}^{F'(N)} E(F_k) \quad (7.1)
\]

Example:
\( N = 12 = 2^2 \cdot 3 \), \( L(N) = 3 \), \( F'(N) = 4 \)

Let \( A_{L(N)} = \{2, 3, 5\} \)

\( F_1 \rightarrow N = 12 = 12 \times 1 \times 1 \), \( x_1 = 2 \), \( x_2 = 1 \)

\[
E(F_1) = \frac{3!}{(2!)(1!)} = 3
\]

\( F_2 \rightarrow N = 12 = 6 \times 2 \times 1 \), \( x_1 = 1 \), \( x_2 = 1 \), \( x_3 = 1 \)

\[
E(F_2) = \frac{3!}{(1!)(1!)(1!)} = 6
\]

\( F_3 \rightarrow N = 12 = 4 \times 3 \times 1 \), \( x_1 = 1 \), \( x_2 = 1 \), \( x_3 = 1 \)

\[
E(F_3) = \frac{3!}{(1!)(1!)(1!)} = 6
\]

\( F_4 \rightarrow N = 12 = 3 \times 2 \times 2 \), \( x_1 = 1 \), \( x_2 = 2 \)

\[
E(F_4) = \frac{3!}{(2!)(1!)} = 3
\]
\[ O(\Psi[N, A_{L(N)}]) = \sum_{k=1}^{F'(N)} E(F_k) = 3 + 6 + 6 + 3 = 18 \]

To verify we have

\[ \Psi[N, A_{L(N)}] = \{ 2^{11}, 3^{11}, 5^{11}, 2^5 \times 3, 2^5 \times 3, 3^5 \times 2, 3^5 \times 5, 5^5 \times 2, \]
\[ 5^5 \times 3, 2^3 \times 3^2, 2^3 \times 5^2, 3^3 \times 2^2, 3^3 \times 5^2, 5^3 \times 2^2, 5^3 \times 3^2, 2^2 \times 3 \times 5, \]
\[ 3^2 \times 2 \times 5, 5^2 \times 2 \times 3, \} \]

**REFERENCES:**


[2] "The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.
A NOTE ON THE SMARANDACHE DIVISOR SEQUENCES

(Anamnath Murthy, S.E. (E&T), Well Logging Services, Oil And Natural Gas Corporation Ltd., Sabarmati, Ahmedabad, India-380005.)

ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION (SFP), as follows:

Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r$ be a set of $r$ natural numbers and $p_1, p_2, p_3, \ldots p_r$ be arbitrarily chosen distinct primes then

$F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ is defined as the number of ways in which the number

$N = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r \overline{p_1 \ p_2 \ p_3 \ldots p_r}$

could be expressed as the product of its' divisors. For simplicity, we denote $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) = F(N)$, where

$N = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r \overline{p_1 \ p_2 \ p_3 \ldots p_r \ldots p_n}$

and $p_r$ is the $r^{th}$ prime. $p_1 = 2$, $p_2 = 3$ etc.

In [2] we have defined SMARANDACHE DIVISOR SEQUENCES as follows

$P_n = \{ x \mid d(x) = n \}$, $d(x) =$ number of divisors of $n$.

$P_1 = \{1\}$

$P_2 = \{ x \mid x$ is a prime $\}$

$P_3 = \{ x \mid x = p^2, \ p$ is a prime $\}$

$P_4 = \{ x \mid x = p^3 \ or \ x = p_1p_2, \ p, p_1, p_2$ are primes $\}$
Let $F_1$ be a SFP of $N$. Let $\Psi_{F_1} = \{ y | d(y) = N \}$, generated by the SFP $F_1$ of $N$. It has been shown in Ref. [3] that each SFP generates some elements of $\Psi$ or $\mathcal{P}_N$. Here each SFP generates infinitely many elements of $\mathcal{P}_N$. Similarly $\Psi_{F_1}$, $\Psi_{F_2}$, $\Psi_{F_3}$, ..., $\Psi_{F_{(N)}}$, are defined. It is evident that all these $F_k$'s are disjoint and also

$$P_N = \cup \Psi_{F_k} \quad 1 \leq k \leq F'(N).$$

**Theorem (7.1)** There are $F'(N)$ disjoint and exhaustive subsets in which $P_N$ can be decomposed.

**Proof:** Let $0 \in P_N$, and let it be expressed in canonical form as follows

$$\theta = \alpha_1 \alpha_2 \cdots \alpha_r \quad \begin{array}{l} \alpha_1 \alpha_2 \cdots \alpha_r \\ p_1 \ p_2 \ p_3 \ldots \ p_r \end{array}$$

Then $d(\theta) = (\alpha_1+1)(\alpha_2+1)(\alpha_3+1) \ldots (\alpha_r+1)$

Hence $\theta \in \Psi_{F_k}$ for some $k$ where $F_k$ is given by

$$N = (\alpha_1+1)(\alpha_2+1)(\alpha_3+1) \cdots (\alpha_r+1)$$

Again if $\theta \in \Psi_{F_s}$, and $\theta \in \Psi_{F_t}$ then from unique factorisation theorem $F_s$ and $F_t$ are identical SFPs of $N$.

**References:**


[4] "The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.
ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION (SFP), as follows:

Let \( \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r \) be a set of \( r \) natural numbers and \( p_1, p_2, p_3, \ldots p_r \) be arbitrarily chosen distinct primes then \( F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) \) called the Smarandache Factor Partition of \( (\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) \) is defined as the number of ways in which the number
\[
N = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r
\]
\( p_1 \ p_2 \ p_3 \ldots p_r \) could be expressed as the product of its' divisors. For simplicity, we denote \( F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) = F'(N) \), where
\[
N = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r \alpha_n
\]
\( p_1 \ p_2 \ p_3 \ldots p_r \ldots p_n \)
and \( p_r \) is the \( r^{th} \) prime. \( p_1 = 2, p_2 = 3 \) etc.

In this note an algorithm to list out all the SFPs of a number without missing any is developed.

DISCUSSION:

DEFINITION: \( F'_x(y) \) is defined as the number of those SFPs of \( y \) which involve terms not greater than \( x \).

If \( F_1 \) be a factor partition of \( y \):

\[
F_1 \longrightarrow x_1 X x_2 X x_3 X \ldots x_r , \text{then } F_1 \text{ is included in } F'_x(y) \text{ iff}
\]
\[ x_i \leq x \quad \text{for} \quad 1 \leq i \leq r \]

Clearly \( F'_x(y) \leq F'(y) \), The equality holds good iff \( x \geq y \).

Example: \( F'_8(24) = 5 \). Out of 7 only the last 5 are included in \( F'_8(24) \).

(1) 24
(2) 12 \times 2
(3) 8 \times 3
(4) 6 \times 4
(5) 6 \times 2 \times 2
(6) 4 \times 3 \times 2
(7) 3 \times 2 \times 2 \times 2.

ALGORITHM: Let \( d_1, d_2, d_3, \ldots, d_r \) be the divisors of \( N \) in descending order. For listing the factor partitions following are the steps:

(A) (1) Start with \( d_1 = N \).
(2) Write all the factor partitions involving \( d_2 \) and so on.
(B) While listing care should be taken that the terms from left to right should be written in descending order.

** At \( d_k \geq N^{1/2} \geq d_{k+1} \), and onwards, step (B) will ensure that there is no repetition.

Example: \( N = 36 \), Divisors are 36, 18, 12, 9, 6, 4, 3, 2, 1.

\[
\begin{align*}
36 &\rightarrow 36 \\
18 &\rightarrow 18 \times 2 \\
12 &\rightarrow 12 \times 3 \\
9 &\rightarrow 9 \times 4 \\
&\quad \quad 9 \times 2 \times 2 \\
6 &\rightarrow 6 \times 6 \\
6 &\rightarrow 6 \times 3 \times 2 \\
\cdots &\cdots \\
4 &\rightarrow 4 \times 3 \times 3
\end{align*}
\]
FORMULA FOR F'(N)

\[ F'(N) = \sum_{d_r|N} F'_{d_r}(N/d_r) \] -------(8.1)

Example:

\[ N = 216 = 2^33^3 \]

(1) \[ 216 \quad \rightarrow F_{216}(1) = 1 \]

(2) \[ 108 \times 2 \quad \rightarrow F_{108}(2) = 1 \]

(3) \[ 72 \times 3 \quad \rightarrow F_{72}(3) = 1 \]

(4) \[ 54 \times 4 \quad \rightarrow F_{54}(4) = 2 \]

(5) \[ 54 \times 2 \times 2 \]

(6) \[ 36 \times 6 \quad \rightarrow F_{36}(6) = 2 \]

(7) \[ 36 \times 3 \times 2 \]

(8) \[ 27 \times 8 \quad \rightarrow F_{27}(8) = 3 \]

(9) \[ 27 \times 4 \times 2 \]

(10) \[ 27 \times 2 \times 2 \times 2 \]

(11) \[ 24 \times 9 \quad \rightarrow F_{24}(9) = 2 \]

(12) \[ 24 \times 3 \times 3 \]

(13) \[ 18 \times 12 \quad \rightarrow F_{18}(12) = 4 \]

(14) \[ 18 \times 6 \times 2 \]

(15) \[ 18 \times 4 \times 3 \]

(16) \[ 18 \times 3 \times 2 \times 2 \]

(17) \[ 12 \times 9 \times 2 \quad \rightarrow F_{12}(18) = 3 \]

(18) \[ 12 \times 6 \times 3 \]

(19) \[ 12 \times 3 \times 3 \times 2 \]

(20) \[ 9 \times 8 \times 3 \quad \rightarrow F_{9}(24) = 5 \]

(21) \[ 9 \times 6 \times 4 \]

(22) \[ 9 \times 6 \times 2 \times 2 \]

(23) \[ 9 \times 4 \times 3 \times 2 \]

(24) \[ 9 \times 3 \times 2 \times 2 \]

(25) \[ 8 \times 3 \times 3 \times 3 \quad \rightarrow F_{8}(27) = 1 \]

(26) \[ 6 \times 6 \times 6 \quad \rightarrow F_{6}(36) = 4 \]

(27) \[ 6 \times 6 \times 3 \times 2 \]

(28) \[ 6 \times 4 \times 3 \times 3 \]

(29) \[ 6 \times 3 \times 3 \times 2 \times 2 \]

(30) \[ 4 \times 3 \times 3 \times 3 \times 2 \times 2 \quad \rightarrow F_{4}(54) = 1 \]

(31) \[ 3 \times 3 \times 3 \times 2 \times 2 \times 2 \quad \rightarrow F_{3}(72) = 1 \]

\[ \quad \rightarrow F_{2}(108) = 0 \]

\[ \quad \rightarrow F_{1}(216) = 0 \]
\[ F'(216) = \sum_{d_r/N} F'_{d_r}(216/d_r) = 31 \]

**Remarks:** This algorithm would be quite helpful in developing a computer program for the listing of SFPs.

**REFERENCES:**


[2] "The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.
EXPANSION OF \( x^n \) IN SMARANDACHE TERMS OF PERMUTATIONS

(Amarnath Murthy, S.E. (E & T), Well Logging Services, Oil And Natural Gas Corporation Ltd., Sabarmati, Ahmedabad, India-380005.)

ABSTRACT:

DEFINITION of SMARANDACHE TERM

Consider the expansion of \( x^n \) as follows

\[
x^n = b(n,1) x + b(n,2) x(x-1) + b(n,3) x(x-1)(x-2) + \ldots + b(n,n) x^n \quad \text{---(9.1)}
\]

We define \( b(n,r) x(x-1)(x-2) \ldots (x-r+1)(x-r) \) as the \( r^{th} \) SMARANDACHE TERM in the above expansion of \( x^n \).

In the present note we study the coefficients \( b(n,r) \) of the \( r^{th} \) SMARANDACHE TERM in such an expansion. We are encountered with fascinating coincidences.

DISCUSSION:

Let us examine the coefficients \( b(n,r) \) of the \( r^{th} \) SMARANDACHE TERM in such an expansion.

Taking \( x = 1 \) gives \( b(n,1) = 1 \)

Taking \( x = 2 \) gives \( b(n,2) = (2^n - 2)/2 \)

Taking \( x = 3 \) gives \( b(n,3) = (3^n - 3 - 6(2^n - 2)/2)/6 \)

\[
= \{1/3!\} \left\{ (1).3^n - (3).2^n + (3)(1)^n - (1)(0)^n \right\}
\]

Taking \( x = 4 \) gives

\[
b(n,4) = (1/4!) \left[ (1)4^n - (4)3^n + (6)2^n - (4)1^n + 1(0)^n \right]
\]
This suggests the possibility of

\[
b(n,r) = \frac{1}{r!} \sum_{k=1}^{r} (-1)^{r-k} \cdot \binom{r}{k} \cdot k^n = a_{n,r}
\]

**THEOREM (9.1)**

\[
b(n,r) = \frac{1}{r!} \sum_{k=1}^{r} (-1)^{r-k} \cdot \binom{r}{k} \cdot k^n = a_{n,r}
\]

**First Proof:**

This will be proved in two parts. First we shall prove the following proposition.

\[
b(n+1,r) = b(n,r-1) + r \cdot b(n,r)
\]

we have

\[
x^n = b(n,1) x + b(n,2) x(x-1) + b(n,3) x(x-1)(x-2) + \ldots + b(n,n) x^n
\]

\[x = r \text{, gives},\]

\[
r^n = b(n,1) r + b(n,2) r(r-1) + b(n,3) r(r-1)(r-2) + \ldots + b(n,n) r^n
\]

multiplying both the sides by \( r \),

\[
r^{n+1} = b(n,1) r \cdot r + b(n,2) r(r-1) + b(n,3) r(r-1)(r-2) + \ldots + b(n,n) r \cdot r
\]

terms equal to zero.

\[
r^{n+1} = b(n,1) r \cdot \binom{r}{1} + b(n,2) r \cdot \binom{r}{2} + b(n,3) r \cdot \binom{r}{3} + \ldots + b(n,n) r \cdot \binom{r}{n}
\]

Using the identity \( r \cdot \binom{r}{k} = \binom{r}{k+1} + k \cdot \binom{r}{k} \) we can write

\[
r^{n+1} = b(n,1) \{ r \cdot \binom{r}{1} + \binom{r}{2} \} + b(n,2) \{ r \cdot \binom{r}{2} + 2 \cdot \binom{r}{3} \} + \ldots + b(n,n) \{ r \cdot \binom{r}{n} + r \cdot \binom{r}{n-1} \}
\]

\[
r^{n+1} = b(n,1) \cdot \binom{r}{1} + \left\{ b(n,1) + 2 \cdot b(n,2) \right\} \cdot \binom{r}{2} + \left\{ b(n,2) + 3 \cdot b(n,3) \right\} \cdot \binom{r}{3} + \ldots +
\]
\{ b(n,r-1) + r \cdot b(n,r) \} \cdot \mathcal{P}_r \quad \text{(9.2)}

Otherwise also we have

\[ r^{n+1} = b_{n+1,1} \cdot \mathcal{P}_1 + b_{n+1,2} \cdot \mathcal{P}_2 + b_{n+1,3} \cdot \mathcal{P}_3 + \ldots + b_{n+1,r} \cdot \mathcal{P}_r \]

The coefficients of \( \mathcal{P}_t \) ( \( t < r \) ) are independent of \( r \) hence these can separately be equated giving us

\[ b_{n+1,r} = b_{n,r-1} + r \cdot b_{n,r} \]

Now we shall proceed by induction. Let

\[ b_{n,r} = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot \binom{r}{k} \cdot k^n \]

\[ b_{n,r-1} = \frac{1}{(r-1)!} \sum_{k=0}^{r-1} (-1)^{r-1-k} \cdot \binom{r-1}{k} \cdot k^n \]

be true. Then the sum \( b_{n,r-1} + r \cdot b_{n,r} \) equals

\[ \frac{1}{(r-1)!} \sum_{k=0}^{r-1} (-1)^{r-1-k} \cdot \binom{r-1}{k} \cdot k^n + r \cdot \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot \binom{r}{k} \cdot k^n \]

\[ = \frac{(-1)^{r-1}}{r!} \left[ \sum_{k=0}^{r-1} (-1)^k \left\{ \binom{r}{k} - \binom{r}{k} \cdot k^n \right\} \right] + r^{n+1}/r! \]

\[ = \frac{(-1)^{r-1}}{r!} \left[ \sum_{k=0}^{r-1} (-1)^k \left\{ -k \cdot \binom{r}{k} \cdot k^n \right\} \right] + r^{n+1}/r! \]

\[ = \frac{1}{r!} \sum_{k=0}^{r-1} (-1)^k \cdot \binom{r}{k} \cdot k^{n+1} \]

which gives us

\[ b_{n+1,r} = \frac{1}{r!} \sum_{k=0}^{r-1} (-1)^k \cdot \binom{r}{k} \cdot k^{n+1} \]

\( b_{n+1,r} \) also takes the same form. Hence by induction the proof is complete.
Second Proof: This proof is totally based on a combinatorial approach. This method also provides us with a proof of the Conjecture (6.3) of ref. [3] as a by product.

If n objects no two alike are to be distributed in x boxes, no two alike, the number of ways this can be done is $x^n$ since there are k alternatives for disposals of the first object, k alternatives for the disposal of the second, and so on.

Alternately let us proceed with a different approach. Let us consider the number of distributions in which exactly a given set of r boxes is filled (rest are empty). Let it be called $f(n,r)$.

We derive a formula for $f(n,r)$ by using the inclusion-exclusion principle. The method is illustrated by the computation of $f(n,5)$. Consider the total number of arrangements, $5^n$ of n different objects in 5 different boxes. Say that such an arrangement has property 'a'. In case the first box is empty, property 'b' incase the second box is empty, and similar property 'c', 'd', and 'e' for the other three boxes respectively. To find the number of distributions with no box empty, we simply count the number of distributions having none of the properties 'a', 'b', 'c', . . . etc. We can apply the following formula.

$$N - \binom{r}{1}N(a) + \binom{r}{2}N(a,b) - \binom{r}{3}N(a,b,c) + \ldots ------(9.3)$$
Here \( N = 5^n \) is the total number of distributions. By \( N(a) \) we mean the number of distributions with the first box empty, and so \( N(a) = 4^n \). Similarly \( N(a,b) \) is the number of distributions with the first two boxes empty. But this is the same as the number of distributions into 3 boxes and \( N(a,b) = 3^n \). Thus we can write

\[
N = 5^n, \quad N(a) = 4^n, \quad N(a,b) = 3^n \quad \text{etc.} \quad N(a,b,c,d,e) = 0.
\]

Applying formula (9.3) we get

\[
f(n,5) = 5^n - \binom{5}{1} 4^n + \binom{5}{2} 3^n - \binom{5}{3} 2^n + \binom{5}{4} 1^n - \binom{5}{5} 0^n
\]

by the direct generalization of this with \( r \) in place of 5, we see that

\[
f(n,r) = r^n - \binom{r}{1} (r-1)^n + \binom{r}{2} (r-2)^n - \binom{r}{3} (r-3)^n + \ldots
\]

\[
f(n,r) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} (r-k)^n
\]

\[
f(n,r) = r! \cdot a_{(n,r)}, \quad \text{from theorem (3.1). of ref. [1]}
\]

Now these \( r \) boxes out of the given \( x \) boxes can be chosen in \( \binom{x}{r} \) ways. Hence the total number of ways in which \( n \) distinct objects distributed in \( x \) distinct boxes occupying exactly \( r \) of them (with the rest \( x-r \) boxes empty), defined as \( d(n,r/x) \), is given by

\[
d(n,r/x) = r! \cdot a_{(n,r)} \binom{x}{r}
\]

\[
d(n,r/x) = a_{(n,r)} \cdot \binom{x}{r}
\]

Summing up all the cases for \( r = 0 \) to \( r = x \), the total number of ways in which \( n \) distinct objects can be distributed in \( x \) distinct boxes is given by

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\[
\sum_{r=0}^{x} d(n,r/x)) = \sum_{r=0}^{x} x^r a_{(n,r)} \tag{9.4}
\]
equating the two results obtained by two different approaches we get
\[
x^n = \sum_{r=0}^{n} x^r a_{(n,r)}
\]

**REMARKS:**
If \(n\) distinct objects are to be distributed in \(x\) distinct boxes with no box empty, then \(n < x\) is mandatory for a possible distribution. e.g. 5 objects cannot be placed in 7 boxes with no empty boxes (a sort of converse of pigeon hole principle).

Hence we get the following result
\[
f(n,r) = 0, \quad \text{for } n < k.
\]
\[
f(n,r) = \sum_{k=0}^{r} (-1)^k rC_k (r-k)^n = 0 \quad \text{if } n < r.
\]

**Further Generalisation:**

(1) One can go ahead with the following generalisation of expansion of \(x^n\) as follows
\[
x^n = g_{(n/k,1)} x + g_{(n/k,2)} x(x-k) + g_{(n/k,3)} x(x-k)(x-2k) + \ldots +
g_{(n/k,n)} x(x-k)(x-2k) \ldots (x-(n-1)k)(x-nk+k)
\]
\[
g_{(n/k,r)} = b_{(n,r)} = a_{(n,r)} \quad \text{for } k = 1 \text{ has been dealt with in this note. One can explore for beautiful patterns for } k = 2, 3 \text{ etc.}
\]
We can call (define) \(g_{(n/k,r)} x(x-k)(x-2k) \ldots (x-(n-1)k)(x-rk+k)\) as the \(r^{th}\) Smarandache Term of the \(k^{th}\) kind in such an
expansion.

(2) Another generalisation could be

\[ x^{n!} = c_{(n/k,1)}(x-k) + c_{(n/k,2)}(x-k)(x^2-k) + c_{(n/k,3)}(x-k)(x^2-k)(x^3-k) + \ldots \]

For \( k = 1 \) if we denote \( c_{(n/k,r)} = c_{(n,r)} \) for simplicity we get

\[ x^{n!} = c_{(n,1)}(x-1) + c_{(n,2)}(x-1)(x^2-1) + c_{(n,3)}(x-1)(x^2-1)(x^3-1) + \ldots \]

We can define \( c_{(n/k,r)}(x-k)(x^2-k)(x^3-k)\ldots(x^n-k) \) as the \( r^{th} \) Smarandache Factorial Term of the \( k^{th} \) kind in the expansion of \( x^{n!} \). One can again explore for patterns for the coefficient \( c_{(n/k,r)} \).

REFERENCES:


ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION (SFP), as follows:

Let \( \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r \) be a set of \( r \) natural numbers and \( p_1, p_2, p_3, \ldots p_r \) be arbitrarily chosen distinct primes then \( F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) \) called the Smarandache Factor Partition of \( (\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) \) is defined as the number of ways in which the number

\[
N = \frac{\alpha_1}{p_1} \frac{\alpha_2}{p_2} \frac{\alpha_3}{p_3} \cdots \frac{\alpha_r}{p_r}
\]

could be expressed as the product of its' divisors. For simplicity, we denote \( F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) = F'(N) \), where

\[
N = \frac{\alpha_1}{p_1} \frac{\alpha_2}{p_2} \frac{\alpha_3}{p_3} \cdots \frac{\alpha_r}{p_r} \frac{\alpha_n}{p_n}
\]

and \( p_r \) is the \( r \)th prime. \( p_1 = 2, p_2 = 3 \) etc.

Also for the case

\[
\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1
\]

we denote

\[
F(1, 1, 1, 1, 1 \ldots) = F(1#n) \leftarrow n \text{- ones} \rightarrow
\]

In [2] we define \( b_{(n,r)} x(x-1)(x-2) \ldots (x-r+1)(x-r) \) as the \( r \)th SMARANDACHE TERM in the expansion of
\[ x^n = b_{(n,1)} x + b_{(n,2)} x(x-1) + b_{(n,3)} x(x-1)(x-2) + \ldots + b_{(n,n)} x^n \]

In this note some more results depicting how closely the coefficients of the SMARANDACHE TERM and SFPs are related are derived.

**DISCUSSION:**

**Result on the \([i^j]\) matrix:**

Theorem (9.1) in [2] gives us the following result

\[ x^n = \sum_{r=0}^{n} x^r a_{(n,r)} \]

which leads us to the following beautiful result.

\[ \sum_{k=1}^{x} k^n = \sum_{k=1}^{x} \sum_{r=1}^{k} k^r a_{(n,r)} \]

In matrix notation the same can be written as follows for \( x = 4 = n \).

\[
\begin{bmatrix}
1_{P1} & 0 & 0 & 0 \\
2_{P1} & 2_{P2} & 0 & 0 \\
3_{P1} & 3_{P2} & 3_{P3} & 0 \\
4_{P1} & 4_{P2} & 4_{P3} & 4_{P4} \\
\end{bmatrix} \ast \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 3 & 7 \\
0 & 0 & 1 & 6 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
1^1 & 1^2 & 1^3 & 1^4 \\
2^1 & 2^2 & 2^3 & 2^4 \\
3^1 & 3^2 & 3^3 & 3^4 \\
4^1 & 4^2 & 4^3 & 4^4 \\
\end{bmatrix}
\]

In general

\[ P \ast A' = Q \]

where \( P = \begin{bmatrix} i^j \end{bmatrix} \)

\( A = \begin{bmatrix} a_{(i,j)} \end{bmatrix} \) and \( Q = \begin{bmatrix} i^j \end{bmatrix} \)

(A' is the transpose of A)

Consider the expansion of \( x^n \), again
\[ x^n = b_{(n,1)} x + b_{(n,2)} x(x-1) + b_{(n,3)} x(x-1)(x-2) + \ldots + b_{(n,n)} x^n \]

for \( x = 3 \) we get

\[ x^3 = b_{(3,1)} x + b_{(3,2)} x(x-1) + b_{(3,3)} x(x-1)(x-2) \]

comparing the coefficient of powers of \( x \) on both sides we get

\[ b_{(3,1)} - b_{(3,2)} + 2 b_{(3,3)} = 0 \]

\[ b_{(3,2)} - 3 b_{(3,3)} = 0 \]

\[ b_{(3,3)} = 1 \]

In matrix form

\[
\begin{bmatrix}
1 & -1 & 2 \\
0 & 1 & -3 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
b_{(3,1)} \\
b_{(3,2)} \\
b_{(3,3)}
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

\[ C_3 \cdot A_3 = B_3 \]

\[ A_3 = C_3^{-1} \cdot B_3 \]

\[ C_3^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ (C_3^{-1})' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \]
similarly it has been observed that

\[
(C_4^{-1})' = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 3 & 1 & 0 \\
1 & 7 & 6 & 1
\end{bmatrix}
\]

The above observation leads to the following theorem.

**THEOREM (10.1)**

In the expansion of \(x^n\) as

\[
x^n = b_{(n,1)} x + b_{(n,2)} x(x-1) + b_{(n,3)} x(x-1)(x-2) + \ldots + b_{(n,n)} x^n
\]

If \(C_n\) be the coefficient matrix of equations obtained by equating the coefficient of powers of \(x\) on both sides then

\[
(C_n^{-1})' = a_{(i,j)} = \text{star matrix of order } n
\]

**PROOF:** It is evident that \(C_{pq}\) the element of the \(p^{th}\) row and \(q^{th}\) column of \(C_n\) is the coefficient of \(x^p\) in \(x^q\). And also \(C_{pq}\) is independent of \(n\). The coefficient of \(x^p\) on the RHS is

\[
\text{coefficient of } x^p = \sum_{q=1}^{n} b_{(n,q)} C_{pq}, \text{ also}
\]

\[
\text{coefficient of } x^p = 1 \text{ if } p = n
\]

\[
\text{coefficient of } x^p = 0 \text{ if } p \neq n.
\]

In matrix notation
\[
\text{coefficient of } x^p = \begin{bmatrix}
\sum_{q=1}^{n} b_{(n,q)} C_{pq} \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\sum_{q=1}^{n} b_{(n,q)} C'_{qp} \\
\end{bmatrix}
\]

\[= i_{np} \text{ where } i_{np} = 1, \text{ if } n = p \text{ and } i_{np} = 0, \text{ if } n \neq p.\]

\[= I_n \text{ (identity matrix of order } n).\]

\[
\begin{bmatrix}
b_{(n,q)} \\
\end{bmatrix}
\begin{bmatrix}
C_{p,q} \\
\end{bmatrix}' = I_n
\]

\[
\begin{bmatrix}
a_{(n,q)} \\
\end{bmatrix}
\begin{bmatrix}
C_{p,q} \\
\end{bmatrix}' = I_n \quad \text{as } b_{(n,q)} = a_{(n,q)}
\]

\[A_n \cdot C_n' = I_n \]

\[A_n = I_n \left[ C_n' \right]^{-1}\]

\[A_n = \left[ C_n' \right]^{-1}\]

This completes the proof of theorem (10.1).
THEOREM (10.2)

If \( C_{k,n} \) is the coefficient of \( x^k \) in the expansion of \( x^P \), then

\[
\sum_{k=1}^{n} F(1\#k) C_{k,n} = 1
\]

PROOF: In property (3) of the STAR TRIANGLE following proposition has been established.

\[ F'(1\#n) = \sum_{m=1}^{n} a_{(n,m)} = B_n \]

in matrix notation the same can be expressed as follows for \( n = 4 \)

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 3 & 7 \\
0 & 0 & 1 & 6 \\
0 & 0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
B_1 & B_2 & B_3 & B_4
\end{bmatrix}
\]

In general

\[
\begin{bmatrix}
1
\end{bmatrix}_{1 \times n} \ast \begin{bmatrix}
a_{(i,j)}
\end{bmatrix}_{n \times n}^{(C_n^{-1})'} = \begin{bmatrix}
B_i
\end{bmatrix}_{1 \times n}
\]

\[
\begin{bmatrix}
B_i
\end{bmatrix}_{1 \times n} \ast \begin{bmatrix}
1
\end{bmatrix}_{n \times n}^{(C_n)} = \begin{bmatrix}
1
\end{bmatrix}_{1 \times n}
\]

In \( C_{n,n} \), \( C_{p,q} \) the \( p \)th row and \( q \)th column is the coefficient of \( x^p \)
in \( x^P \). Hence we have
\[ \sum_{k=1}^{n} F(1\#k) C_{k,n} = 1 = \sum_{k=1}^{n} B_k C_{k,n} \]

**THEOREM (10.3)**

\[ \sum_{k=1}^{n} F(1\#(k+1)) C_{k,n} = n + 1 = \sum_{k=1}^{n} B_{k+1} C_{k,n} \]

**PROOF:**

It has already been established that

\[ B_{n+1} = \sum_{m=1}^{n} (m+1) a_{n,m} \]

In matrix notation

\[
\begin{bmatrix}
  j+1 \\
  1 \times n
\end{bmatrix} 
\begin{bmatrix}
  a_{(i,j)} \\
  1 \times n
\end{bmatrix} ' = 
\begin{bmatrix}
  B_{j+1} \\
  1 \times n
\end{bmatrix} 
\begin{bmatrix}
  c_n^{-1} \\
  n \times n
\end{bmatrix}
\]

\[
\begin{bmatrix}
  j+1 \\
  1 \times n
\end{bmatrix} = 
\begin{bmatrix}
  B_{j+1} \\
  1 \times n
\end{bmatrix} * 
\begin{bmatrix}
  c_n \\
  n \times n
\end{bmatrix}
\]

\[ \sum_{k=1}^{n} B_{k+1} C_{k,n} = n + 1 \]

There exist ample scope for more such results.

**REFERENCES:**


[5] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.
SMARANDACHE MAXIMUM RECIPROCAL REPRESENTATION FUNCTION

(Amarnath Murthy, S.E. (E &T), Well Logging Services, Oil And Natural Gas Corporation Ltd., Sabarmati, Ahmedbad, India- 380005.)

ABSTRACT: Smarandache Maximum Reciprocal Representation (SMRR) Function \( f_{\text{SMRR}}(n) \) is defined as follows

\[
f_{\text{SMRR}}(n) = t \text{ if } \sum_{r=1}^{t} \frac{1}{r} \leq n \leq \sum_{r=1}^{t+1} \frac{1}{r}
\]

SMARANDACHE MAXIMUM RECIPROCAL REPRESENTATION SEQUENCE

SMRRS is defined as \( T_n = f_{\text{SMRR}}(n) \)

\[
f_{\text{SMRR}}(1) = 1
\]

\[
f_{\text{SMRR}}(2) = 3, \quad (1 + \frac{1}{2} + \frac{1}{3} < 2 < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4})
\]

\[
f_{\text{SMRR}}(3) = 10 \quad \sum_{r=1}^{10} \frac{1}{r} \leq \sum_{r=1}^{11} \frac{1}{r} \leq \sum_{r=1}^{11} \frac{1}{r}
\]

SMRRS is

\[
1, 3, 10, \ldots
\]

A note on The SMRR Function:

The harmonic series \( \sum \frac{1}{n} \) satisfies the following inequality

\[
\log (n+1) < \sum \frac{1}{n} < \log n + 1 \quad -------(1)
\]
This inequality can be derived as follows. We have 
\[ e^x > 1 + x > \quad x > 0 \]
and 
\[ (1 + 1/n)^{(1 + 1/n)} > 1 \quad n > 0 \]
which gives
\[ 1/(r+1) < \log(1 + 1/r) < 1/r \]
summing up for \( r = 1 \) to \( n+1 \) and with some algebraic jugglery
we get (1). With the help of (1) we get the following result on the SMRR function.

If SMRR(n) = m then \([\log(m)] \approx n - 1\)
Where \([\log(m)]\) stands for the integer value of \(\log(m)\).

**SOME CONJECTURES:**

(1.1). Every positive integer can be expressed as the sum of the reciprocal of a finite number of distinct natural numbers. ( in infinitely many ways.).

Let us define a function \( R_m(n) \) as the minimum number of natural numbers required for such an expression.

(1.2). Every natural number can be expressed as the sum of the reciprocals of a set of natural numbers which are in Arithmetic Progression.

(1.3). Let
\[ \sum 1/r \leq n \leq \sum 1/(r+1) \]
where \( \sum 1/r \) stands for the sum of the reciprocals of first \( r \)
natural numbers and let $S_1 = \sum 1/r$

let $S_2 = S_1 + 1/(r+k_1)$ such that $S_2 + 1/(r+k_1+1) > n \geq S_2$

let $S_3 = S_2 + 1/(r+k_2)$ such that $S_3 + 1/(r+k_2+1) > n \geq S_3$

and so on, then there exists a finite $m$ such that

$S_{m+1} + 1/(r+k_m) = n$

Remarks: The veracity of conjecture (1.1) is deducible from conjecture (1.3).

(1.4). (a) There are infinitely many disjoint sets of natural numbers sum of whose reciprocals is unity.

(b) Among the sets mentioned in (a), there are sets which can be organised in an order such that the largest element of any set is smaller than the smallest element of the next set.

REFERENCES:


[3] "The Florentine Smarandache" Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.
(1.1) To derive a formula for SFPs of given length $m$ of $p^aq^a$ for any value of $a$.

(1.2) To derive a formula for SFPs of

$$N = p_1^{2} p_2^{2} p_3^{2} \ldots p_r^{2}$$

(1.3) To derive a formula for SFPs of given length $m$ of

$$N = p_1^{a} p_2^{a} p_3^{a} \ldots p_r^{a}$$

(1.4) To derive a reduction formula for $p^aq^a$ as a linear combination of $p^{a-r}q^{a-r}$ for $r = 0$ to $a-1$.

Similar reduction formulae for (1.2) and (1.3) also.

(1.5) In general, in how many ways a number can be expressed as the product of its divisors?

(1.6) Every positive integer can be expressed as the sum of the reciprocal of a finite number of distinct natural numbers. (in infinitely many ways.)

Let us define a function $R_m(n)$ as the minimum number of natural numbers required for such an expression.
(1.7). Every natural number can be expressed as the sum of the reciprocals of a set of natural numbers which are in Arithmetic Progression.

(1.8). Let
\[ \sum \frac{1}{r} \leq n \leq \sum \frac{1}{r+1} \]
where \( \sum \frac{1}{r} \) stands for the sum of the reciprocals of first \( r \) natural numbers and let \( S_1 = \sum \frac{1}{r} \)
let \( S_2 = S_1 + \frac{1}{(r+k_1)} \) such that \( S_2 + \frac{1}{(r+k_1+1)} > n \geq S_2 \)
let \( S_3 = S_2 + \frac{1}{(r+k_2)} \) such that \( S_3 + \frac{1}{(r+k_2+1)} > n \geq S_3 \)
and so on, then there exists a finite \( m \) such that
\[ S_{m+1} + \frac{1}{(r+k_m)} = n \]

Remarks: The veracity of conjecture (1.6) is deducible from conjecture (1.8).

(1.9). (a) There are infinitely many disjoint sets of natural numbers sum of whose reciprocals is unity.
(b) Among the sets mentioned in (a), there are sets which can be organised in an order such that the largest element of any set is smaller than the smallest element of the next set.

DEFINITION: We can define Smarandache Factor Partition Sequence as follows: \( T_n = \) factor partition of \( n = F'(n) \)
\[ T_1 = 1, T_2 = 3, T_{12} = 4 \text{ etc.} \]
SFPS is given by
\[1, 1, 1, 2, 1, 2, 1, 3, 2, 2, 1, 4, 1, 2, 2, 5, 1, 4, 1, 4, 2, 2, 1, 7, 2, \ldots,\]

**DEFINITION:** Let \( S \) be the smallest number such that \( F'(S) = n \). We define \( S \) a **Vedam Number** and the sequence formed by Vedam numbers as the **Smarandache Vedam Sequence**.

Smarandache Vedam Sequence is given as follows: \( T_n = F'(S) \)
\[1, 4, 8, 12, 16, \ldots, 24, \ldots\]

**Note:** There exist no number whose factor partition is equal to 6, hence a question mark at the sixth slot. We define such numbers as **Dull numbers**. The readers can explore the distribution (frequency) and other properties of dull numbers.

**DEFINITION:** A number \( n \) is said to be a **Balu number** if it satisfies the relation \( d(n) = F'(n) = r \), and is the smallest such number.

1, 16, 36 are all Balu numbers.
\[d(1) = F'(1) = 1 \quad d(16) = F'(16) = 5 \quad d(36) = F'(36) = 9.\]

Each Balu number \( \geq 16 \), generates a **Balu Class** \( C_B(n) \) of numbers having the same canonical form satisfying the equation \( d(m) = F'(m) \). e.g. \( C_B(16) = \{ x \mid x = p^4, \ p \text{ is a prime.} \} = \{ 16, 81, 256, \ldots \} \). Similarly \( C_B(36) = \{ x \mid x = p^2q^2, \ p \text{ and } q \text{ are primes.} \} \)
Conjecture

(1.10): There are only finite number of Balu Classes.

In case Conjecture (1.10) is true, to find out the largest Balu number.

REFERENCES


[8] "The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.
SMARANDACHE RECIPROCAL FUNCTION 
AND AN ELEMENTARY INEQUALITY

( Amarnath Murthy, S.E.(E&T), Well Logging Services, 
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The Smarandache Function is defined as \( S(n) = k \). Where \( k \) is the 
smallest integer such that \( n \) divides \( k! \).

Let us define \( S_c(n) \) Smarandache Reciprocal Function as follows:

\[ S_c(n) = x \text{ where } x + 1 \text{ does not divide } n! \text{ and for every } y \leq x, \ y \mid n! \]

THEOREM-I.

If \( S_c(n) = x \), and \( n \leq 3 \), then \( x + 1 \) is the smallest prime greater than \( n \).

PROOF: It is obvious that \( n! \) is divisible by \( 1, 2, \ldots \) up to \( n \). We have 
to prove that \( n! \) is also divisible by \( n + 1, n + 2, \ldots n + m (= x) \), where 
\( n + m + 1 \) is the smallest prime greater than \( n \). Let \( r \) be any of these 
composite numbers. Then \( r \) must be factorable into two factors each of 
which is \( \geq 2 \). Let \( r = p.q \), where \( p, q \geq 2 \). If one of the factors (say \( q \)) is 
\( \geq n \) then \( r = p.q \geq 2n \). But according to the Bertrand's postulate there 
must be a prime between \( n \) and \( 2n \), there is a contradiction here since all 
the numbers from \( n + 1 \) to \( n + m \) \(( n + 1 \leq r < n + m )\) are assumed to be 
composite. Hence neither of the two factors \( p, q \) can be \( \geq n \). So each must 
be \( < n \). Now there are two possibilities:
Case-I \( p \neq q \).

In this case as each is <\( n \) so \( p.q = r \) divides \( n! \)

Case-II \( p = q = \text{prime} \)

In this case \( r = p^2 \) where \( p \) is a prime. There are again three possibilities:

(a) \( p \geq 5 \). Then \( r = p^2 > 4p \Rightarrow 4p < r < 2n \Rightarrow 2p < n \). Therefore both \( p \) and \( 2p \) are less than \( n \) so \( p^2 \) divides \( n! \)

(b) \( p = 3 \), Then \( r = p^2 = 9 \) Therefore \( n \) must be 7 or 8. and 9 divides 7! and 8!.

(c) \( p = 2 \), then \( r = p^2 = 4 \). Therefore \( n \) must be 3. But 4 does not divide 3!, Hence the theorem holds for all integral values of \( n \) except \( n = 3 \). This completes the proof.

Remarks: Readers can note that \( n! \) is divisible by all the composite numbers between \( n \) and \( 2n \).

Note: We have the well known inequality \( S(n) \leq n \). \( \text{(2)} \)

From the above theorem one can deduce the following inequality.

If \( p_r \) is the \( r^{\text{th}} \) prime and \( p_r \leq n < p_{r+1} \) then \( S(n) \leq p_r \). (A slight improvement on (2)).

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i.e. \( S(p_r) = p_r, S(p_r + 1) < p_r, S(p_r + 2) < p_r, \ldots S(p_{r+1} - 1) < p_r, S(p_{r+1}) = p_{r+1} \)

Summing up for all the numbers \( p_r \leq n < p_{r+1} \) one gets

\[
\sum_{t=0}^{p_{r+1} - p_r - 1} S(p_r + t) \leq (p_{r+1} - p_r) p_r
\]

Summing up for all the numbers up to the \( s^{th} \) prime, defining \( p_0 = 1 \), we get

\[
\sum_{k=1}^{p_s} S(k) \leq \sum_{r=0}^{s} (p_{r+1} - p_r) p_r \quad \text{(3)}
\]

More generally from Ref. [1] following inequality on the \( n \)th partial sum of the Smarandache (Inferior) Prime Part Sequence directly follows.

**Smarandache (Inferior) Prime Part Sequence**

For any positive real number \( n \) one defines \( p_p(n) \) as the largest prime number less than or equal to \( n \). In [1] Prof. Krassimir T. Atanassov proves that the value of the \( n^{th} \) partial sum of this sequence \( X_n = \sum_{k=1}^{n} p_p(k) \) is given by

\[
X_n = \sum_{k=2}^{\pi(n)} (p_k - p_{k-1}) \cdot p_{k-1} + (n - p_{\pi(n)} + 1) \cdot p_{\pi(n)} \quad \text{(4)}
\]

From (3) and (4) we get
\[ \sum_{k=1}^{n} S(k) \leq X_n \]

REFERENCES:


[2] "The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.

ABSTRACT: A number is said to be a Smarandache Lucky Number if an incorrect calculation leads to a correct result. For example, the fraction \( \frac{42}{21} = \frac{(4-2)}{(2-1)} = \frac{21}{1} = 2 \) is incorrectly calculated, but the result 2 is still correct. More generally a Smarandache Lucky Method is said to be any incorrect method which leads to a correct result. In Ref. [1] the following question is asked:

(1) Are there infinitely many Smarandache Lucky Numbers?

We claim that the answer is YES.

Also in the present note we give some fascinating Smarandache Lucky Methods in algebra, trigonometry, and calculus.

A SMARANDACHE LUCKY METHOD IN TRIGONOMETRY:

Some students at the early stage of just having introduced to the concept of function, misunderstand the meaning of \( f(x) \) as the product of \( f \) and \( x \).

E.g. for them \( \sin(x) = \) product of \( \sin \) and \( x \). This gives rise to a funny lucky method applicable to the following identity.

To prove
\[
\sin^2(x) - \sin^2(y) = \sin(x + y) \cdot \sin(x - y)
\]

\[
\text{LHS} = \sin^2(x) - \sin^2(y)
\]

\[
= \{\sin(x) + \sin(y)\} \cdot \{\sin(x) - \sin(y)\} \quad \text{----------(A)}
\]
Taking \( \text{sin} \) common from both the factors

\[
= \{ \text{sin} \ (x + y) \} \cdot \{ \text{sin} \ (x - y) \}
\]

= RHS

The correct method from (A) onwards should have been

\[
= \{2 \text{sin}((x + y)/2). \text{cos}((x - y)/2)\} \cdot \{2 \text{cos}((x + y)/2). \text{sin}((x - y)/2)\}. \\
= \{2 \text{sin}((x + y)/2). \text{cos}((x + y)/2)\} \cdot \{2 \text{cos}((x - y)/2) \text{sin}((x - y)/2)\}. \\
= \{ \text{sin} \ (x + y) \} \cdot \{ \text{sin} \ (x - y) \}
\]

= RHS

Remarks: The funny thing is the wrong lucky method is a shortcut more so to get carried away.

A SMARANDACHE LUCKY METHOD IN ALGEBRA:

In vector algebra the dot product of two vectors \((a_1 \text{i} + a_2 \text{j} + a_3 \text{k})\) and \((b_1 \text{i} + b_2 \text{j} + b_3 \text{k})\) is given by

\[
(a_1 \text{i} + a_2 \text{j} + a_3 \text{k}) \cdot (b_1 \text{i} + b_2 \text{j} + b_3 \text{k}) = a_1b_1 + a_2b_2 + a_3b_3
\]

The same idea if extended to ordinary algebra would mean

\[
(a + b) (c + d) = ac + bd. \quad \text{----------}(B)
\]

This wrong lucky method is applicable in proving the following algebraic identity.

\[
a^3 + b^3 + c^3 - 3abc = (a + b + c) (a^2 + b^2 + c^2 - ab - bc - ca)
\]

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RHS = (a + b + c) \( (a^2 + b^2 + c^2 - ab - bc - ca) \)

= (a + b + c) \{ (a^2 - bc) + (b^2 - ac) + (c^2 - ab) \}

applying the wrong lucky method (B), one gets

= a.(a^2 - bc) + b.(b^2 - ac) + c.(c^2 - ab)

= a^3 - abc + b^3 - abc + c^3 - abc

= a^3 + b^3 + c^3 - 3 abc = LHS

A SMARANDACHE LUCKY METHOD IN CALCULUS:

The fun involved in the following lucky method in calculus is two fold. It goes like this. A student is asked to differentiate the product of two functions. Instead of applying the formula for differentiation of product of two functions he applies the method of integration of the product of two functions (Integration by parts) and gets the correct answer. The height of coincidence is if another student given the same product of two functions and asked to integrate does the reverse of it i.e. he ends up in applying the formula for differentiation of the product of two functions and yet gets the correct answer. I would take the liberty to call such a lucky method to be Smarandache superlucky method. The suspense ends.

Consider the product of two functions x and sin(x).

\( f(x) = x \) and \( g(x) = \sin(x) \)

The Smarandache lucky method of differentiation (integration by parts) is

\[
d\{ f(x).g(x) \}/dx = f(x) \int g(x)dx - \int [\{d(f(x))/dx\}. \int g(x)dx]dx
\]

\[
d\{(x).\sin(x)\}/dx = (x) \int \sin(x)dx - \int [\{d(x)/dx\}. \int \sin(x)dx]dx
\]
\[ = - (x) \cdot (\cos(x)) + \sin(x) \]
\[ = - x \cdot \cos(x) + \sin(x) \]

The Smarandache lucky method of Integration

\[ \int \{f(x) \cdot g(x)\} \, dx = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x) \]

Consider the same functions again we get by this lucky method

\[ \int \{x \cdot \sin(x)\} \, dx = (x) \cdot \{ \cos(x) \} + \{ \sin(x) \} \quad (1) \]

Or

\[ \int \{x \cdot \sin(x)\} \, dx = x \cdot \cos(x) + \sin(x) \]

That, both the answers are correct, can be verified, by applying the right methods.

REFERENCE:


[2] 'F.SMARANDACHE', Funny Problems, Special Collections, Arizona State University, Hayden Library, Tempe, AZ, USA.
Review of


As a collection of diverse persons, mathematicians suffer from more negative stereotypes than almost any other group. This is unfortunate, discouraging and most often wrong. Widely characterized as lacking in humor, abstract and considered to be brilliant, eccentric imbeciles by much of the public, mathematicians rarely fit that description. Of course, branding a group with a stereotype is often a mask for insecurities. Ralph P. Boas Jr. is a fascinating counterexample to most of these inaccurate assumptions. Filled with humor, verse and mathematics, his optimism and love of life are captured just like the lions so prominently featured in the book.

So, how does an unarmed person capture a lion using only the weapons of mathematical thought? There are more ways than you would think. Over thirty different "proven" methods are given. My favorite is: "The lion is big game, hence certainly a game. There exists an optimal strategy. Follow it." It seems that every area of mathematics can be used to construct a way to capture a lion. Of course, some are more efficient than others.

The verse varies from limericks to some that were seeded by material from Shakespeare. All are quite good, although it is necessary to read some of them twice in order to capture the intended meaning. Most mathematicians have heard of Nicolas Bourbaki, the mathematical polyglot who is in fact a pseudonym for a collection of French mathematicians. When it came time to publish the first material on the mathematics of lion hunting, Boas and his colleagues chose the pseudonym, Hector Petard, from the Shakespearean line, "the engineer, hoist with his own petard"; Hamlet Act III, Scene IV. To complete the circle, Boas and friends also "arranged" for a wedding between Betti Bourbaki and H. Petard and duly announced the upcoming event.

Another main section of the book consists of reminiscences by Boas and those who knew him best. As a mathematical man of mischief and an educator, he had few equals. Several short papers describing some of his basic ideas for education are also included. These ideas share one common trait. Simple to understand and execute. No fancy or complex methods, just fundamental strategies to make mathematics more understandable.

The final part of the book consists of short anecdotes about his experiences in mathematics. Some are about fellow mathematicians, others about students and the rest about whatever seemed to happen during his eventful life. At times amusing, other times profound, but at all times interesting, they are simple notes describing how the mathematical world works.

Despite common misconceptions, there are some mathematicians who contain a bit of the sprite and Ralph P. Boas Jr. was such a person. That impishness is captured in this book, which is reason enough to read it.
Review of


Originally published in 1942, this book has lost none of its power in the last half century.
It is a commentary on the recent demise of geometry in many curricula that 33 years
elapsed between the publication of the fifth and sixth editions. Fortunately, like so many
things in the world, trends in mathematics are cyclic, and one can hope that the geometric
cycle is on the rise. We in mathematics owe so much to geometry. It is generally
conceded that much of the origins of mathematics is due to the simple necessity of
maintaining accurate plots in settlements. The only book from the ancient history of
mathematics that all mathematicians have heard of is the Elements by Euclid. It is one of
the most read books of all time, arguably the only book without a religious theme still in
widespread use over 2000 years after the publication of the first edition. The geometry
taught in high schools today is with only minor modifications found in the Euclidean
classic.

There are other reasons why geometry should occupy a special place in our hearts. Most of the
principles of the axiomatic method, the concept of the theorem and many of the techniques used
in proofs were born and nurtured in the cradle of geometry. For many centuries, it was nearly an
act of faith that all of geometry was Euclidean. That annoying fifth postulate seemed so out of
place and yet it could not be made to go away. Many tried to remove it, but finally the Holmesean
dictum of, "once you have eliminated the impossible, what is left, not matter how improbable,
must be true", had to be admitted. There were in fact three geometries, all of which are of equal
validity. The other two, elliptic and hyperbolic, are the main topics of this wonderful book.

Coxeter is arguably the best geometer of this century but there can be no argument that he is the
best explainer of geometry of this century. While fifty years is a mere spasm compared to the
time since Euclid, it is certainly possible that students will be reading Coxeter far into the future
with the same appreciation that we have when we read Euclid. His explanations of the
non-Euclidean geometries is so clear that one cannot help but absorb the essentials. In so many
ways, Euclidean geometry is but the middle way between the two other geometries. A point well
made and in great detail by Coxeter.

Geometry is a jewel that was born on the banks of the Nile river and we should treasure and
respect it as the seed from which so much of our basic reasoning processes sprouted. For this
reason, you should buy this book and keep a copy on your shelf.

Reviewed by

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This attractively, but inexpensively, produced monograph introduces the reader to the flavour of the problems proposed in recent years by Smarandache and others. These are generally of two kinds: those which deal with recurring patterns within sequences and those which extend classical number theoretic results. Those who are familiar with the author's style will recognise his ingenuity in the latter and his tenacity in the former.

An example of the former is Smarandache Problem 22: "In the sequence of Smarandache Square Complements: 
{1,2,3,1,5,6,7,2,1,10,11,3,14,15,1,17,2,19,5, 
21,22,23,6,1,26,3,7,29,30,31,2,33,34,35,1,37, 
38,39,10,41,42,43,11,5,46,47,3,1,2,51,13,53,6,55, 
14,57,58,59,15,61,62,7,1,65,66,67,17,69,70,71,2,...), for each integer n find the smallest integer k such that nk is a perfect square."

An extension of a classical result is Smarandache Problem 117: "Let p be an odd positive number. Prove that p and p+2 are twin primes iff
\[(p-1)!\{(1/p)+(2/(p+2))\}+(1/p)+(1/(p+2))\] is an integer."

Atanassov develops and utilises properties of "new" functions such as the inverse factorial function defined by: x? = y iff y! =x, and his digit sum which has appeared in a number of papers over the last decade. These ensure that his solutions are always elegant rather than the result of brute force. Readers might like to try their hands at the above two problems and then buy the book to enjoy more of these problems which are easy to understand but not always easy to solve.

The book is in microfilm format too, and can be ordered from: UMI, PO Box 1346, Ann Arbor, MI 48106-1346, USA; tel: 1-800-521-0600.

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