A Generalization of a Theorem of Carnot

**Theorem of Carnot:** Let $M$ be a point on the diagonal $AC$ of an arbitrary quadrilateral $ABCD$. Through $M$ one draws a line which intersects $AB$ in $\alpha$ and $BC$ in $\beta$. Let us draw another line, which intersects $CD$ in $\gamma$ and $AD$ in $\delta$. Then one has:

\[
\frac{A\alpha}{B\alpha} \cdot \frac{B\beta}{C\beta} \cdot \frac{C\gamma}{D\gamma} \cdot \frac{D\delta}{A\delta} = 1.
\]

**Generalization:** Let $A_1, A_2, A_3, ..., A_n$ be a polygon. On a diagonal $A_i A_k$ of this polygon one takes a point $M$ through which one draws a line $d_1$ which intersects the lines $A_1 A_2, A_2 A_3, ..., A_{k-1} A_k$ respectively in the points $P_1, P_2, ..., P_{k-1}$ and another line $d_2$ intersects the other lines $A_{k+1} A_{k+2}, A_{k+2} A_{k+3}, ..., A_{n-1} A_n$ respectively in the points $P_{k+1}, ..., P_{n-1}, P_n$. Then one has:

\[
\prod_{i=1}^{n} \frac{A_i P_i}{A_{\varphi(i)} P_i} = 1,
\]

where $\varphi$ is the circular permutation

\[
\begin{pmatrix}
1 & 2 & \cdots & n-1 & n \\
2 & 3 & \cdots & n & 1
\end{pmatrix}.
\]

**Proof:**

Let us have $1 \leq j \leq k-1$. One easily shows that:

\[
\frac{A_j P_j}{A_{j+1} P_j} = D(A_j, d_1) \frac{D(A_j, d_1)}{D(A_{j+1}, d_1)}
\]

where $D(A, d)$ represents the distance from the point $A$ to the line $d$, since the triangles $P_j A_j A_{j+1}$ and $P_j A_{j+1} A_{j+1}$ are similar. (One notes with $A_j'$ and $A_j''$ the projections of the points $A_j$ and $A_{j+1}$ on the line $d_1$).

It results from it that:

\[
\frac{A_1 P_1}{A_2 P_1} \frac{A_2 P_2}{A_3 P_2} \cdots \frac{A_{k-1} P_{k-1}}{A_k P_{k-1}} = D(A_1, d_1) \frac{D(A_1, d_1)}{D(A_2, d_1)} \frac{D(A_2, d_1)}{D(A_3, d_1)} \cdots \frac{D(A_{k-1}, d_1)}{D(A_k, d_1)} = \frac{D(A_1, d_1)}{D(A_k, d_1)}
\]

In a similar way, for $k \leq h \leq n$ one has:

\[
\frac{A_h P_h}{A_{\varphi(h)} P_h} = \frac{D(A_h, d_2)}{D(A_{\varphi(h)}, d_2)}
\]

and

\[
\prod_{h=k}^{n} \frac{A_h P_h}{A_{\varphi(h)} P_h} = \frac{D(A_k, d_2)}{D(A_1, d_2)}
\]
The product of the theorem is equal to:

\[
\frac{D(A_1, d_1)}{D(A_k, d_1)} \cdot \frac{D(A_k, d_2)}{D(A_1, d_2)},
\]

but

\[
\frac{D(A_1, d_1)}{D(A_k, d_1)} = \frac{A_1 M}{A_k M}
\]

since the triangles \( MA_i A_i \) and \( MA_k A_k \) are similar. In the same way, because the triangles \( MA_i A_i' \) and \( MA_k A_k' \) are similar (one notes with \( A_i' \) and \( A_k' \) the respective projections of \( A_i \) and \( A_k \) on the line \( d_2 \)), one has:

\[
\frac{D(A_k, d_2)}{D(A_1, d_2)} = \frac{A_k M}{A_1 M}.
\]

The product from the statement is therefore equal to 1.

Remark: If one replaces \( n \) by 4 in this theorem, one finds the theorem of Carnot.