Almost Unbiased Estimator for Estimating Population Mean Using Known Value of Some Population Parameter(s)

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Abstract

In this paper we have proposed an almost unbiased estimator using known value of some population parameter(s). Various existing estimators are shown particular members of the proposed estimator. Under simple random sampling without replacement (SRSWOR) scheme the expressions for bias and mean square error (MSE) are derived. The study is extended to the two phase sampling. Empirical study is carried out to demonstrate the superiority of the proposed estimator.

Key words: Auxiliary information, bias, mean square error, unbiased estimator, two phase sampling.

1. Introduction

Consider a finite population \( U = U_1, U_2, ... U_N \) of \( N \) units. Let \( y \) and \( x \) stand for the variable under study and auxiliary variable respectively. Let \((y_i, x_i), i=1, 2, ..., n\) denote the values of the units included in a sample \( s_n \) of size \( n \) drawn by simple random sampling without replacement (SRSWOR). The auxiliary information has been used in improving the precision of the estimate of a parameter (See Cochran (1977), Sukhatme et. al. (1984) and the references cited there in). Out of many methods, ratio and product methods of estimation are good illustrations in this context.

In order to have a survey estimate of the population mean \( \bar{y} \) of the study character \( y \), assuming the knowledge of the population mean \( \bar{x} \) of the auxiliary character \( x \), the well-known ratio estimator is

\[
t_r = \bar{y} \frac{x}{\bar{x}}
\]  

(1.1)

Bahl and Tuteja (1991) suggested an exponential ratio type estimator as –

\[
t_{re} = \bar{y} \exp \left[ \frac{\bar{x} - x}{\bar{x} + x} \right]
\]

(1.2)
Several authors have used prior value of certain population parameter(s) to find more precise estimates. Sisodiya and Dwivedi (1981), Sen (1978) and Upadhyaya and Singh (1984) used the known coefficient of variation (CV) of the auxiliary character for estimating population mean of a study character in ratio method of estimation. The use of prior value of coefficient of kurtosis in estimating the population variance of study character was first made by Singh et. al. (1973). Later used by Singh and Kakaran (1993) in the estimation of population mean of study character. Singh and Tailor (2003) proposed a modified ratio estimator by using the known value of correlation coefficient. Kadilar and Cingi (2006), Khosnevisan et. al. (2007), Singh et. al. (2007) Singh and Kumar (2009) and Singh et. al. (2009) have suggested modified ratio estimators by using different pairs of known value of population parameter(s).

In this paper under SRSWOR, we have proposed almost unbiased estimator for estimating $Y$.

2. Almost unbiased ratio type estimator

Suppose

$$t_0 = \bar{y}, \quad t_{rs} = \bar{y} \left( \frac{a\bar{x}+b}{x+b} \right), \quad t_{rse} = \bar{y} \exp \left\{ \frac{(a\bar{x}+b)-\left(\frac{a\bar{x}+b}{x+b}\right)}{\left(\frac{a\bar{x}+b}{x+b}\right) + (a\bar{x}+b)} \right\}$$

Such that $t_0, t_{rs}, t_{rse} \in w_r$, where $w_r$ denotes the set of all possible ratio type estimators for estimating the population mean $\bar{Y}$. By definition the set $w_r$ is a linear variety, if

$$t_{wr} = \omega_0\bar{y} + \omega_1 t_{rs} + \omega_2 t_{rse} \in w$$

for $\sum_{i=0}^{2} \omega_i = 1, \quad \omega_i \in \mathbb{R}$ (2.1)

where $\omega_i (i=0, 1, 2)$ denotes the statistical constants and $\mathbb{R}$ denotes the set of real numbers.

To obtain the bias and MSE of $t_{wr}$, we write

$$\bar{y} = \bar{Y}(1 + e_0), \quad \bar{x} = \bar{X}(1 + e_1),$$

such that

$$E(0) = E(e_1) = 0.$$

$$E(e_0^2) = f_1 C_y^2, \quad E(e_1^2) = f_1 C_x^2, \quad E(e_0 e_1) = f_1 \rho C_y C_x.$$

where $f_1 = \left( \frac{1}{n} - \frac{1}{N} \right)$,

$$S_y^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} (y_i - \bar{Y})^2, \quad S_x^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} (x_i - \bar{X})^2,$$

$$C_y = \frac{S_y}{\bar{Y}}, \quad C_x = \frac{S_x}{\bar{X}}, \quad K = \rho \left( \frac{C_y}{C_x} \right), \quad \rho = \frac{S_{yx}}{S_y S_x},$$

$$S_{yx} = \frac{1}{(N-1)} \sum_{i=1}^{N} (y_i - \bar{Y})(x_i - \bar{X}).$$
Expressing $t_w$ in terms of $e$'s, we have

$$t_w = \bar{y}(1 + e_0)[\omega_0 + \omega_1 (1 + \theta e_1)^{-1} + \omega_2 \exp \left\{ -\frac{\theta e_1}{2} (1 + \theta e_1)^{-1} \right\}]$$

(2.3)

where $\theta = \frac{s}{s^2 + b}$.

Expanding the right hand side of (2.3) and retaining terms up to second order of $e$'s, we have

$$t_w \simeq \bar{y} \left[ 1 + e_0 - \omega \theta e_1 + \theta^2 (\omega_1 + \frac{3}{8} \omega_2) e_1^2 - \theta \omega e_0 e_1 \right]$$

(2.4)

where $\omega = (\omega_1 + \frac{\omega_2}{2})$.

Taking expectations of both side of (2.4) and then subtracting $\bar{y}$ from both side, we get the bias of the estimator $t_w$, up to the first order of approximation as

$$B(t_w) = f_1 \bar{y} \left[ \theta^2 c_x^2 (\omega_1 + \frac{3}{8} \omega_2) - \theta \omega \rho c_y c_x \right]$$

(2.5)

$$B(t_{rs}) = f_1 \bar{y} [\theta^2 c_x^2 - \theta \rho c_y c_x]$$

(2.6)

$$B(t_{rse}) = f_1 \bar{y} \left[ \frac{3 \theta^2 c_x^2}{8} - \frac{\theta \rho c_y c_x}{2} \right]$$

(2.7)

From (2.4), we have

$$t_w - \bar{y} \simeq \bar{y} [e_0 - \theta \omega e_1]$$

(2.8)

Squaring both sides of (2.9) and then taking expectations, we get MSE of the estimator $t_w$, up to the first order of approximation, as

$$MSE(t_w) = f_1 \bar{y} \left[ c_y^2 + \theta^2 \omega^2 c_x^2 - 2 \theta \omega \rho c_y c_x \right]$$

(2.9)

This is minimum when

$$\omega = k (= \rho \frac{c_y}{c_x})$$

(2.10)

Putting this value of $\omega (= k)$ in (2.10), we get the minimum MSE of $t_w$ as

$$MSE(t_w) = f_1 \bar{y}^2 c_y^2 (1 - \rho^2)$$

(2.11)

which is same as that of traditional linear regression estimator from (2.5) and (2.11), we have

$$\omega_1 + \frac{\omega_2}{2} = k.$$  

(2.12)

From (2.2) and (2.13), we have only two equations in three unknowns. It is not possible to find the unique values for $\omega_i$, $i = 0, 1, 2$. In order to get unique values for $\omega_i$, we shall impose the linear restriction

$$\omega_0 B(\bar{y}) + \omega_1 B(t_{rs}) + \omega_2 B(t_{rse}) = 0$$

(2.13)

Equations (2.2), (2.11) and (2.14) can be written as in the matrix form as

$$\begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 1 & \frac{1}{2} \\ 0 & B(t_{rs}) & B(t_{rse}) \end{bmatrix} \begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} k \end{bmatrix}$$

(2.14)
Using (2.15) ,we get unique values of $\omega_i$ ($i=0,1,2$) as

$$
\begin{align*}
\omega_0 &= \frac{\Delta_0}{\Delta_r} \\
\omega_1 &= \frac{\Delta_1}{\Delta_r} \\
\omega_2 &= \frac{\Delta_2}{\Delta_r} 
\end{align*}
$$

(2.16)

where

$$
\Delta_r = B(t_{rse}) - \frac{1}{2} B(t_{se})
$$

$$
\Delta_{r0} = B(t_{rse})[1 - k] + \frac{k}{2} B(t_{se}) \left( k - \frac{1}{2} \right)
$$

(2.17)

Use of these $\omega_i$ ($i=0,1,2$) remove the bias up to terms of order ($n^{-1}$) at (2.1).

3. **Product –type estimators**

Suppose $t_0 = \bar{y}$, $t_{ps} = \bar{y} \left( \frac{aX+b}{aX+b} \right)$, $t_{pse} = \bar{y} \exp \left[ (aX+b)-(aX+b) \right]$, such that $t_0$, $t_{ps}$, $t_{pse} \in Q$, where $Q$ denotes the set of all possible product –type estimators for estimating the population mean $\bar{Y}$. By definition, the set $Q$ is linear variety if

$$
\sum_{i=0}^{2} q_i = 1, \quad q_i \in R
$$

(3.2)

where $q_i$ ($i=0,1,2$) denotes the statistical constants.

Expressing $t_q$ in terms of e’s, we have

$$
t_q = \omega_0 \bar{Y} + \omega_1 (1 + \theta e_1) + \omega_2 \exp \left[ \frac{q_0}{2} (1 + \theta e_1)^{-1} \right] \tag{3.3}
$$

where $\theta = \frac{aX+b}{aX+b}$.

Expanding the right hand side of (3.3) and retaining terms up to second power of e’s, we have

$$
t_q \approx \bar{Y} \left[ 1 + e_0 + \theta q_2 e_1 - \frac{q_0}{2} e_1^2 + q \theta e_0 e_1 \right] \tag{3.4}
$$

where $q = q_1 + \frac{q_2}{2}$

(3.5)

Taking expectations of both sides of (3.4) and then subtracting $\bar{Y}$ from both sides, we get the bias of the estimator $t_q$, up to the first order of approximation as

$$
B(t_q) = f_1 \bar{Y} \left[ -\frac{q_0}{8} \theta^2 C_x^2 + q \theta p C_y C_x \right] \tag{3.6}
$$
Bias expression for the estimators $t_{ps}$ and $t_{pse}$ is given by

$$B(t_{ps}) = f_1 \frac{1}{Y} \left[ \theta p C_y C_x \right]$$

$$B(t_{pse}) = f_1 \frac{1}{Y} \left[ -\frac{1}{8} \theta^2 C_y^2 + \frac{\theta p C_y C_x}{2} \right]$$

From (3.4), we have

$$(t_q - \bar{Y}) \approx \bar{Y}[e_0 + \theta q e_1]$$

Squaring both the sides of (3.9) and then taking expectations, we get MSE of the estimator $t_q$, up to the first order of approximation, as

$$\text{MSE}(t_q) = f_1 \frac{1}{Y^2} \left[ C_y^2 + \theta^2 q^2 C_x^2 + 2\theta q p C_y C_x \right]$$

which is minimum for

$$q = -k = -\frac{C_y}{C_x}$$

Putting this value of $q(-k)$ in (3.10), we get the minimum MSE of $t_q$ as

$$\text{min. MSE}(t_q) = f_1 \frac{1}{Y^2} C_y^2 (1 - p^2)$$

which is same as that of traditional linear regression estimator.

From (3.5) and (3.11), we have

$$q_1 + \frac{3\theta}{2} = -k$$

From (3.2) and (3.13), we have only two equations in three unknowns. It is not possible to find the unique values for $q_i$'s, $i=0,1,2$. In order to get unique values of $q_i$'s, we shall impose the linear restriction

$$q_0 B(\bar{Y}) + q_1 B(t_{ps}) + q_2 B(t_{pse}) = 0$$

Equations (3.2),(3.13) and (3.14) can be written in the matrix form as

$$\begin{bmatrix}
1 & 1 & 1/2 \\
0 & 1 & 1 \\
0 & B(t_{ps}) & B(t_{pse})
\end{bmatrix}
\begin{bmatrix}
q_0 \\
q_1 \\
q_2
\end{bmatrix}
= \begin{bmatrix}
1 \\
-k \\
0
\end{bmatrix}$$

Solving (3.15), we get the unique values of $q_i$'s ($i=0,1,2$) as-

$$q_0 = \frac{\Delta p}{\Delta_p}$$

$$q_1 = \frac{\Delta p^2}{\Delta_p}$$

$$q_2 = \frac{\Delta p^3}{\Delta_p}$$

where
Use of these q’s (i=0,1,2) remove the bias up to terms of order o(n^{-1}) at (3.1). In Appendix A we have listed some of the important known estimators of the population mean, which can be obtained by suitable choice of constants \( \omega_i, q_i, i=0,1,2 \) and a and b.

### 4. Proposed estimators in two phase sampling

When \( \bar{X} \) is unknown, it is sometimes estimated from a preliminary large sample of size \( n' \) on which only the characteristic \( x \) is measured (for details see Singh et al. (2007)). Then a second phase sample of size \( n (n < n') \) is drawn on which both \( y \) and \( x \) characteristics are measured. Let \( \bar{x} = \frac{1}{n'} \sum_{i=1}^{n'} x_i \) denote the sample mean of \( x \) based on first phase sample of size \( n' \), \( \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i \) and \( \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \) be the sample means of \( y \) and \( x \) respectively based on second phase of size \( n \).

In two phase sampling the estimator \( t_{wr} \) will take the following form

\[
t_{wd} = \omega_{0d} \bar{y} + \omega_{1d} t_{rse} + \omega_{2d} t_{rse} \in \omega_d
\]

for \( \sum_{i=0}^{2} \omega_{id} = 1 \), \( \omega_{id} \in \mathbb{R} \) \( (4.1) \)

where \( t_{rsg} = \frac{\bar{y}}{(aX+b)} \) and \( t_{rse} = \frac{\bar{y} \exp \left( \frac{(aX+b)-(aX+b)}{(aX+b)+(aX+b)} \right)}{2} \)

To obtain the bias and MSE of \( t_{wd} \), we write

\[
\bar{y} = \bar{Y}(1 + e_0), \quad \bar{x} = \bar{X}(1 + e_1), \quad \bar{x}' = \bar{X}(1 + e'_1)
\]

such that

\[
E(e_0) = E(e_1) = E(e'_1) = 0.
\]

\[
E(e_0^2) = f_1 C_y^2, \quad E(e_1^2) = f_1 C_x^2, \quad E(e'_1^2) = f_2 C_x^2, \quad E(e_0 e_1) = f_1 \rho C_y C_x
\]

\[
E(e_0 e'_1) = f_2 \rho C_y C_x, \quad E(e_1 e'_1) = f_2 C_x^2
\]

where

\[
f_1 = \left( \frac{1}{n} - \frac{1}{N} \right), \quad f_2 = \left( \frac{1}{n} - \frac{1}{N} \right)
\]

Following the procedure mentioned in section 2 and 3, we get bias and MSE of \( t_{wd} \) as

\[
B(t_{wd}) = \bar{y} \left[ \theta^2 C_x^2 f_3 \left( \omega_{1d} + \frac{3 \omega_{2d}}{8} \right) - \theta \rho C_y C_x f_5 \left( \omega_{1d} + \frac{\omega_{2d}}{2} \right) \right]
\]

(4.3)
Almost Unbiased Estimator for Estimating Population Mean using Known Value of Some Population Parameter(s)

\[
\text{MSE}(t_{\omega_d}) = \sum_{i=1}^{2} \left[ \frac{f_1 C_i^2 + f_2 C_i^2}{n} \omega_i^2 - 2f_3 \rho C_i C_{\omega_d} \omega_i \right]
\]

(4.4)

where \( f_3 = \frac{1}{n} - \frac{1}{n} = f_1 - f_2 \).

MSE \( (t_{\omega_d}) \) is minimum, when

\[
\omega_{1d} + \frac{\omega_{2d}}{2} = \omega_d = k
\]

(4.5)

Putting this value of \( \omega_d \) in (4.4), we get the minimum MSE of \( t_{\omega_d} \) as

\[
\text{min. MSE}(t_{\omega_d}) = \sum_{i=1}^{2} \left[ \frac{f_1 - f_3 \rho^2}{n} \right]
\]

(4.6)

This is same as that of traditional two phase linear regression estimator.

The bias expression for the estimators \( t_{\text{rad}} \) and \( t_{\text{rde}} \) is respectively given by

\[
B(t_{\text{rad}}) = \sum_{i=1}^{2} \left[ \theta^2 C_i^2 f_3 \omega_{1d} - \theta \rho C_i C_{\omega_{rad}} f_3 \omega_{1d} \right]
\]

(4.7)

\[
B(t_{\text{rde}}) = \sum_{i=1}^{2} \left[ \theta^2 C_i^2 f_3 \frac{3}{2} \omega_{2d} - \theta \rho C_i C_{\omega_{rde}} f_3 \omega_{2d} \right]
\]

(4.8)

From (4.2) and (4.5), we have only two equations in three unknowns. It is not possible to find the unique values for \( w_{id} \)'s \( i=0,1,2 \).

In order to get unique values of \( w_{id} \), we shall impose linear restriction

\[
\omega_{0d} B(\bar{y}) + \omega_{1d} B(t_{\text{rad}}) + \omega_{2d} B(t_{\text{rde}}) = 0
\]

(4.9)

Equations (4.2), (4.5) and (4.9) can be written in matrix form, as

\[
\begin{bmatrix}
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\omega_{0d} \\
\omega_{1d} \\
\omega_{2d}
\end{bmatrix} = \begin{bmatrix}
0 \\
B(t_{\text{rad}}) \\
B(t_{\text{rde}})
\end{bmatrix}
\]

(4.10)

Solving (4.10), we get the unique values of \( \omega_{id} \), \( i = 0,1,2 \) as

\[
\begin{cases}
\omega_{0d} = \Delta_{0d} \\
\omega_{1d} = \Delta_{1d} \\
\omega_{2d} = \Delta_{2d}
\end{cases}
\]

(4.11)

where

\[
\begin{align*}
\Delta_{0d} &= B(t_{\text{rde}}) - \frac{1}{2} B(t_{\text{rad}}) \\
\Delta_{1d} &= B(t_{\text{rad}}) \left[ 1 - k \right] + \frac{1}{2} B(t_{\text{rde}}) \left[ k - \frac{1}{2} \right] \\
\Delta_{2d} &= k B(t_{\text{rde}})
\end{align*}
\]

(4.12)

Use of these \( w_{id} \)'s \( i=0,1,2 \) will remove the bias up to terms of order \( O(n^{-1}) \) at (4.1).
The estimator $t_q$ written in (3.1), in two phase sampling, will take following form

$$t_{qd} = q_{0d} \bar{y} + q_{1d} t_{psd} + q_{2d} t_{psde} \in Q_e$$

(4.13)

For $\sum^2_{i=0} q_{id} = 1, \ q_{id} \in \mathbb{R}$

(4.14)

where $q_{id}$'s $(i=0,1,2)$ denotes the statistical constants.

The estimators $t_{psd}$ and $t_{psde}$ are

$$t_{psd} = \bar{y} \left( \frac{aX+b}{aX+b} \right) \quad \text{and}$$

$$t_{psde} = \bar{y} \exp \left\{ \frac{(a\bar{X}+b)-(a\bar{X}+b)}{(a\bar{X}+b)+(a\bar{X}+b)} \right\}$$

Following the procedure of section 4, we get the unique values of $q_{id}$'s $(i=0,1,2)$ as

$$q_{0d} = \frac{\Delta_{p0d}}{\Delta_{pd}}$$

$$q_{1d} = \frac{\Delta_{p1d}}{\Delta_{pd}}$$

$$q_{2d} = \frac{\Delta_{p2d}}{\Delta_{pd}}$$

(4.15)

where

$$\Delta_{pd} = B(t_{psde}) - \frac{1}{2} B(t_{psd})$$

$$\Delta_{p0d} = B(t_{psed}) \{1 + k\} + \frac{1}{2} B(t_{psd}) \{-k - \frac{1}{2}\}$$

$$\Delta_{p1d} = -k \cdot B(t_{psde})$$

$$\Delta_{p2d} = k \cdot B(t_{psd})$$

(4.16)

where

$$B(t_{psd}) = \bar{y} \left[ \theta q_{1d} f_3 \rho C_y C_x \right]$$

(4.17)

$$B(t_{psde}) = \bar{y} \left[ \theta \frac{q_{1d}}{2} f_3 \rho C_y C_x - \frac{\theta^2}{8} q_{2d} f_3 C_y^2 \right]$$

(4.18)

The minimum MSE of $t_{qd}$ is given by

$$\text{MSE}(t_{qd}) = \bar{y}^2 C_y^2 \left[ f_1 - f_3 \rho^2 \right]$$

5. Empirical study

For empirical study we use the data sets earlier used by Kadilar and Cingi (2006) (population 1) and Khosnevisan et. al. (2007) (population 2) to verify the theoretical results.
Almost Unbiased Estimator for Estimating Population Mean using Known Value of Some Population Parameter(s)

Data statistics

<table>
<thead>
<tr>
<th>Population</th>
<th>N</th>
<th>n</th>
<th>( \bar{y} )</th>
<th>( \bar{x} )</th>
<th>( c_\gamma )</th>
<th>( c_\alpha )</th>
<th>( \rho )</th>
<th>( \beta_2(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population 1</td>
<td>106</td>
<td>20</td>
<td>2212.59</td>
<td>27421.7</td>
<td>5.22</td>
<td>2.10</td>
<td>0.86</td>
<td>34.57</td>
</tr>
<tr>
<td>Population 2</td>
<td>20</td>
<td>8</td>
<td>19.55</td>
<td>18.8</td>
<td>0.355</td>
<td>0.394</td>
<td>-0.92</td>
<td>3.06</td>
</tr>
</tbody>
</table>

Table 5.1: Values of \( \omega \)'s and \( q_i \)'s

<table>
<thead>
<tr>
<th>( \omega )'s</th>
<th>Population 1</th>
<th>Population 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_0 )</td>
<td>8.590718</td>
<td>7.892148</td>
</tr>
<tr>
<td>(( q_0 ))</td>
<td>(21.417)</td>
<td>(2.919085)</td>
</tr>
<tr>
<td>( \omega_1 )</td>
<td>11.86615</td>
<td>5.234461</td>
</tr>
<tr>
<td>(( q_1 ))</td>
<td>(16.14158)</td>
<td>(3.576773)</td>
</tr>
<tr>
<td>( \omega_2 )</td>
<td>-19.4569</td>
<td>-12.1266</td>
</tr>
<tr>
<td>(( q_2 ))</td>
<td>(-36.5586)</td>
<td>(-5.49586)</td>
</tr>
</tbody>
</table>

The percent relative efficiencies (PRE) of various estimators of \( \bar{Y} \) are computed and presented in Table 5.2 below.

Table 5.2: PRE of different estimators of \( \bar{Y} \)

<table>
<thead>
<tr>
<th>Estimator</th>
<th>PRE (Pop I)</th>
<th>Estimator</th>
<th>PRE (Pop II)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_0 )</td>
<td>100</td>
<td>( t_0 )</td>
<td>100</td>
</tr>
<tr>
<td>( t_1 )</td>
<td>212.816</td>
<td>( q_0 )</td>
<td>526</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>212.803</td>
<td>( q_1 )</td>
<td>550.261</td>
</tr>
<tr>
<td>( t_3 )</td>
<td>212.606</td>
<td>( q_2 )</td>
<td>645.256</td>
</tr>
<tr>
<td>( t_4 )</td>
<td>212.815</td>
<td>( q_3 )</td>
<td>534.592</td>
</tr>
<tr>
<td>( t_5 )</td>
<td>212.716</td>
<td>( q_4 )</td>
<td>581.732</td>
</tr>
<tr>
<td>( t_6 )</td>
<td>212.810</td>
<td>( q_5 )</td>
<td>465.501</td>
</tr>
<tr>
<td>( t_7 )</td>
<td>143.992</td>
<td>( q_6 )</td>
<td>384.447</td>
</tr>
<tr>
<td>( t_8 )</td>
<td>143.923</td>
<td>( q_7 )</td>
<td>285.920</td>
</tr>
<tr>
<td>( t_9 )</td>
<td>143.988</td>
<td>( q_8 )</td>
<td>338.487</td>
</tr>
<tr>
<td>( t_{10} )</td>
<td>143.990</td>
<td>( q_{10} )</td>
<td>374.584</td>
</tr>
<tr>
<td>( t_{11} )</td>
<td>143.991</td>
<td>( q_{11} )</td>
<td>345.118</td>
</tr>
<tr>
<td>( t_{12} )</td>
<td>143.959</td>
<td>( q_{12} )</td>
<td>231.602</td>
</tr>
<tr>
<td>( t_{13} )</td>
<td>143.991</td>
<td>( q_{13} )</td>
<td>424.194</td>
</tr>
<tr>
<td>( t_{14} )</td>
<td>143.987</td>
<td>( q_{14} )</td>
<td>360.086</td>
</tr>
<tr>
<td>( t_{15} )</td>
<td>143.992</td>
<td>( q_{15} )</td>
<td>356.520</td>
</tr>
<tr>
<td>( t_{16} )</td>
<td>143.911</td>
<td>( q_{16} )</td>
<td>467.051</td>
</tr>
<tr>
<td>( t_{w(\text{opt})} )</td>
<td>384.025</td>
<td>( t_{q(\text{opt})} )</td>
<td>650.263</td>
</tr>
</tbody>
</table>
Rajesh Singh, Mukesh Kumar, Florentin Smarandache

In order to see the performance of the suggested estimators in two phase sampling we use the data set of Murthy (1967) (Population III) and Steel and Torrie (1960) (Population IV).

<table>
<thead>
<tr>
<th>Population</th>
<th>$C_y$</th>
<th>$C_x$</th>
<th>$\rho$</th>
<th>$N$</th>
<th>$n'$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population 1</td>
<td>0.3542</td>
<td>0.9484</td>
<td>0.9150</td>
<td>80</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>Population 2</td>
<td>0.4803</td>
<td>0.7493</td>
<td>-0.4996</td>
<td>30</td>
<td>12</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 5.3: The values of $\omega_{id}'s$ and $q_{id}'s$

<table>
<thead>
<tr>
<th>$\omega_{id}'s$ (q_{id}'s)</th>
<th>Population III</th>
<th>Population IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_{0d}$</td>
<td>-0.241523</td>
<td>3.011435</td>
</tr>
<tr>
<td>$(q_{0d})$</td>
<td>(1.808833)</td>
<td>(1.089979)</td>
</tr>
<tr>
<td>$\omega_{1d}$</td>
<td>-0.558071</td>
<td>1.370950</td>
</tr>
<tr>
<td>$(q_{1d})$</td>
<td>(0.125381)</td>
<td>(0.730464)</td>
</tr>
<tr>
<td>$\omega_{2d}$</td>
<td>1.799595</td>
<td>-3.382385</td>
</tr>
<tr>
<td>$(q_{2d})$</td>
<td>(-0.934214)</td>
<td>(-0.820443)</td>
</tr>
</tbody>
</table>

The percent relative efficiencies of various estimators of $\bar{Y}$ in two phase sampling are computed and presented in Table 5.4 below.

Table 5.4: PRE of different estimators of $\bar{Y}$ in two phase sampling

<table>
<thead>
<tr>
<th>Estimator</th>
<th>PRE (Population I)</th>
<th>PRE (Population II)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{Y}$</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>$t_{rad}$</td>
<td>36.642</td>
<td>24.562</td>
</tr>
<tr>
<td>$t_{psd}$</td>
<td>4.849</td>
<td>59.770</td>
</tr>
<tr>
<td>$t_{red}$</td>
<td>200.420</td>
<td>48.365</td>
</tr>
<tr>
<td>$t_{psd}$</td>
<td>23.628</td>
<td>115.142</td>
</tr>
<tr>
<td>$t_{wd}$</td>
<td>276.156</td>
<td>63.452</td>
</tr>
<tr>
<td>$t_{qd}$</td>
<td>34.321</td>
<td>123.762</td>
</tr>
<tr>
<td>$t_{opt}$</td>
<td>276.156</td>
<td>123.762</td>
</tr>
</tbody>
</table>

6. Conclusion

From theoretical discussion and empirical study we conclude that the proposed estimators under optimum conditions perform better than other estimators considered in the article. The relative efficiencies of various estimators are listed in Table 5.2 and 5.4.
Appendix A

Table A.1: Some members of the proposed family of estimators -

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>( \omega_0 ) (q_0)</th>
<th>( \omega_1 ) (q_1)</th>
<th>( \omega_2 ) (q_2)</th>
<th>Ratio Estimator (corresponding to ( \omega_i ), i=0,1,2)</th>
<th>Product Estimator (corresponding to ( q_i ), i=0,1,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( t_0 = \bar{y} )</td>
<td>( q_0 = \bar{y} )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( t_1 = \bar{y} \frac{\bar{X}}{\bar{X}} )</td>
<td>( q_1 = \bar{y} \frac{\bar{X}}{\bar{X}} )</td>
</tr>
<tr>
<td>1</td>
<td>( c_x )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( t_2 = \bar{y} \frac{\bar{X} + \bar{C}_x}{\bar{X} + \bar{C}_x} )</td>
<td>( q_2 = \bar{y} \frac{\bar{X} + \bar{C}_x}{\bar{X} + \bar{C}_x} )</td>
</tr>
<tr>
<td>1</td>
<td>( \beta_2(x) )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( t_3 = \bar{y} \frac{\bar{X} + \beta_2(x)}{\bar{X} + \beta_2(x)} )</td>
<td>( q_3 = \bar{y} \frac{\bar{X} + \beta_2(x)}{\bar{X} + \beta_2(x)} )</td>
</tr>
<tr>
<td>( \beta_2(x) )</td>
<td>( c_x )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( t_4 = \bar{y} \frac{\bar{X} \beta_2(x) + \bar{C}_x}{\bar{X} \beta_2(x) + \bar{C}_x} )</td>
<td>( q_4 = \bar{y} \frac{\bar{X} \beta_2(x) + \bar{C}_x}{\bar{X} \beta_2(x) + \bar{C}_x} )</td>
</tr>
<tr>
<td>( c_x )</td>
<td>( \beta_2(x) )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( t_5 = \bar{y} \frac{\bar{X} \beta_2(x) + \bar{C}_x}{\bar{X} \beta_2(x) + \bar{C}_x} )</td>
<td>( q_5 = \bar{y} \frac{\bar{X} \beta_2(x) + \bar{C}_x}{\bar{X} \beta_2(x) + \bar{C}_x} )</td>
</tr>
<tr>
<td>1</td>
<td>( \rho )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( t_6 = \bar{y} \left[ \frac{\bar{X} + \rho}{\bar{X} + \rho} \right] )</td>
<td>( q_6 = \bar{y} \left[ \frac{\bar{X} + \rho}{\bar{X} + \rho} \right] )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( t_7 = \bar{y} \exp \left[ \frac{\bar{X} - \bar{X}}{\bar{X} + \bar{X}} \right] )</td>
<td>( q_7 = \bar{y} \exp \left[ \frac{\bar{X} - \bar{X}}{\bar{X} + \bar{X}} \right] )</td>
</tr>
</tbody>
</table>
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\text{a} & \text{b} & \omega_0 (q_0) & \omega_1 (q_1) & \omega_2 (q_2) & \text{Ratio Estimator (corresponding to } \omega_i,i=0,1,2) & \text{Product Estimator (corresponding to } q_i,i=0,1,2) \\
\hline
1 & \beta_2 (x) & 0 & 0 & 1 & t_8 = \hat{y} \exp \left[ \frac{X - \bar{x}}{X + \bar{x} + 2\beta_2 (x)} \right] & q_8 = \hat{y} \exp \left[ \frac{X - \bar{x}}{X + \bar{x} + 2\beta_2 (x)} \right] \\
\hline
1 & c_x & 0 & 0 & 1 & t_9 = \hat{y} \exp \left[ \frac{X - \bar{x}}{X + \bar{x} + 2c_x} \right] & q_9 = \hat{y} \exp \left[ \frac{X - \bar{x}}{X + \bar{x} + 2c_x} \right] \\
\hline
1 & \rho & 0 & 0 & 1 & t_{10} = \hat{y} \exp \left[ \frac{X - \bar{x}}{X + \bar{x} + 2\rho} \right] & q_{10} = \hat{y} \exp \left[ \frac{X - \bar{x}}{X + \bar{x} + 2\rho} \right] \\
\hline
\beta_2 (x) & c_x & 0 & 0 & 1 & t_{11} = \hat{y} \exp \left[ \frac{\beta(x) (\bar{X} - \bar{x})}{\beta(x) (\bar{X} + \bar{x}) + 2c_x} \right] & q_{11} = \hat{y} \exp \left[ \frac{\beta(x) (\bar{X} - \bar{x})}{\beta(x) (\bar{X} + \bar{x}) + 2c_x} \right] \\
\hline
c_x & \beta_2 (x) & 0 & 0 & 1 & t_{12} = \hat{y} \exp \left[ \frac{c_x (\bar{X} - \bar{x})}{c_x (\bar{X} + \bar{x}) + 2\beta_2 (x)} \right] & q_{12} = \hat{y} \exp \left[ \frac{c_x (\bar{X} - \bar{x})}{c_x (\bar{X} + \bar{x}) + 2\beta_2 (x)} \right] \\
\hline
c_x & \rho & 0 & 0 & 1 & t_{13} = \hat{y} \exp \left[ \frac{c_x (\bar{X} - \bar{x})}{c_x (\bar{X} + \bar{x}) + 2\rho} \right] & q_{13} = \hat{y} \exp \left[ \frac{c_x (\bar{X} - \bar{x})}{c_x (\bar{X} + \bar{x}) + 2\rho} \right] \\
\hline
\rho & c_x & 0 & 0 & 1 & t_{14} = \hat{y} \exp \left[ \frac{\rho (\bar{X} - \bar{x})}{\rho (\bar{X} + \bar{x}) + 2c_x} \right] & q_{14} = \hat{y} \exp \left[ \frac{\rho (\bar{X} - \bar{x})}{\rho (\bar{X} + \bar{x}) + 2c_x} \right] \\
\hline
\beta_2 (x) & \rho & 0 & 0 & 1 & t_{15} = \hat{y} \exp \left[ \frac{\beta_2 (x) (\bar{X} - \bar{x})}{\beta_2 (x) (\bar{X} + \bar{x}) + 2\rho} \right] & q_{15} = \hat{y} \exp \left[ \frac{\beta_2 (x) (\bar{X} - \bar{x})}{\beta_2 (x) (\bar{X} + \bar{x}) + 2\rho} \right] \\
\hline
\rho & \beta_2 (x) & 0 & 0 & 1 & t_{16} = \hat{y} \exp \left[ \frac{\rho (\bar{X} - \bar{x})}{\rho (\bar{X} + \bar{x}) + 2\beta_2 (x)} \right] & q_{16} = \hat{y} \exp \left[ \frac{\rho (\bar{X} - \bar{x})}{\rho (\bar{X} + \bar{x}) + 2\beta_2 (x)} \right] \\
\hline
\end{tabular}
\end{table}
Almost Unbiased Estimator for Estimating Population Mean using Known Value of Some Population Parameter(s)

In addition to above estimators a large number of estimators can also be generated from the proposed estimators just by putting different values of constants \( \omega_i, q_i = 0, 1, 2 \), \( a \) and \( b \).

Acknowledgements

The second author (Mukesh Kumar) is thankful to UGC, New Delhi, India, for providing financial assistance. The authors would like to thank the referee for his constructive suggestions on an earlier draft of the paper.

References


