Three Non-linear $\alpha$-Discounting MCDM-Method Examples

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Abstract.
In this paper we present three new examples of using the $\alpha$-Discounting Multi-Criteria Decision Making Method in solving non-linear problems involving algebraic equations and inequalities in the decision process.

Introduction.
We have defined [see 1] a new procedure called $\alpha$-Discounting Method for Multi-Criteria Decision Making ($\alpha$-DMCDM), which is an alternative and extension of Saaty’s Analytical Hierarchy Process (AHP). We have also defined the degree of consistency (and implicitly a degree of inconsistency) of a decision-making problem [1]. The $\alpha$-D MCDM can deal with any set of preferences that can be transformed into an algebraic system of linear and/or non-linear homogeneous and/or non-homogeneous equations and/or inequalities.
We discuss below three new examples of non-linear decision making problems.

Example 1.
The Set of References is $\{C_1, C_2, C_3\}$.
1. $C_1$ is as important as the product of $C_2$ and $C_3$.
2. The square of $C_2$ is as important as $C_3$.
3. $C_3$ is less important than $C_2$.
We denote $C_1 = x$, $C_2 = y$, $C_3 = z$, and we’ll obtain the following non-linear algebraic system of two equations and one inequality:

$$\begin{align*}
x &= yz \\
y^2 &= z \\
z &< y
\end{align*}$$

and of course the conditions

$$\begin{align*}
x + y + z &= 1 \\
x, y, z &\in [0,1]
\end{align*}$$

From the first two equations we have:

$$x = yz = y \cdot y^2 = y^3$$
and 

\[ z = y^2 \]

whence the general priority vector is 
\[ \langle y^2, y, y^2 \rangle. \]

We consider \( y \neq 0 \), because if \( y = 0 \) the priority vector becomes \( \langle 0, 0, 0 \rangle \) which does not make sense.

Dividing by \( y \) we have \( \langle y^2, 1, y \rangle \), and normalized:
\[
\begin{pmatrix}
\frac{y^2}{y^2 + y + 1} & \frac{1}{y^2 + y + 1} & \frac{y}{y^2 + y + 1}
\end{pmatrix}.
\]

From \( y^2 = z \) and \( z < y \)
we have \( y^2 < y \) or \( y^2 - y < 0 \) or \( y(y-1) < 0 \),

hence \( y \in (0,1) \).

For \( y \in (0,1) \) we have the order:
\[
\frac{1}{y^2 + y + 1} > \frac{y}{y^2 + y + 1} > \frac{y^2}{y^2 + y + 1}
\]
so 
\[
C_2 > C_3 > C_1.
\]

**Example 2.**
The Set of References is also \( \{ C_1, C_2, C_3 \} \).

1. \( C_1 \) is as important as the square of \( C_2 \).
2. \( C_2 \) is as important as double \( C_3 \).
3. The square of \( C_3 \) is as important as triple \( C_1 \).

We again denote \( C_1 = x \), \( C_2 = y \), \( C_3 = z \), and we’ll obtain the following non-linear algebraic system of three equations:
\[
\begin{align*}
x &= y^2 \\
y &= 2z \\
z^2 &= 3x.
\end{align*}
\]

If we solve it, we get:
\[
\begin{align*}
x &= 4z^2 \\
z^2 &= 3y \\
x &= \frac{z^2}{3}
\end{align*}
\Rightarrow 4z^2 = \frac{z^2}{3} \Rightarrow z = 0 \Rightarrow y = 0 \Rightarrow x = 0.
\]

Algebraically the only solution is \( \langle 0, 0, 0 \rangle \), but the null solution is not convenient for MCDM.

Let’s parameterize, i.e. “discount” each equality:
\[
\begin{aligned}
\begin{cases}
x = \alpha_1 y^2 \\
y = 2\alpha_2 z \\
z^2 = 3\alpha_3 x
\end{cases}
\Rightarrow x = \alpha_1 \left(2\alpha_2 z\right)^2 = 4\alpha_1\alpha_2^2 z^2
\Rightarrow 4\alpha_1\alpha_2^2 = \frac{1}{3\alpha_3}
\end{aligned}
\]

or
\[12\alpha_1\alpha_2^2\alpha_3 = 1,
\]

which is the characteristic parametric equation needed for the consistency of this algebraic system.

The general algebraic solution in this parameterized case is:
\[<4\alpha_1\alpha_2^2 z^2 2\alpha_2 z z>.
\]

Using the Fairness Principle as in the \(\alpha\)-Discounting MCDM Method, we set all parameters equal:
\[\alpha_1 = \alpha_2 = \alpha_3 = \alpha
\]

whence, from the characteristic parametric equation, we obtain that
\[12\alpha^4 = 1,
\]

therefore
\[\alpha = \frac{1}{\sqrt[4]{12}} = \frac{1}{\sqrt{12}} \approx 0.537.
\]

Thus the general solution for the Fairness Principle is:
\[<4\alpha^3 z^2 2\alpha z z>.
\]

and, after substituting \(\alpha \approx 0.537\), it results:
\[
\begin{cases}
x = 4\alpha^3 z^2 = 0.619z^2 \\
y = 2\alpha z = 1.074z.
\end{cases}
\]

Whence the general solution for the Fairness Principle becomes:
\[<4\alpha^3 z^2 2\alpha z z> = <0.619z^2 1.074z z>.
\]

and dividing by \(z \neq 0\) one has:
\[<0.619z, 1.074, 1>.
\]

But \(y = 1.074 > 1 = z\), hence \(y > z\).

Discussion:

a) If \(z < \frac{1}{0.619} \approx 1.616\), then \(y > z > x\).

b) If \(z = \frac{1}{0.619} \approx 1.616\), then \(y > z = x\).

c) If \(\frac{1}{0.619} < z < \frac{1.074}{0.619}\) or \(1.616 < z < 1.735\), then \(y > x > z\).

d) If \(z = \frac{1.074}{0.619} \approx 1.735\), then \(x = y > z\).
e) If \( z > \frac{1.074}{0.619} \approx 1.735 \), then \( x > y > z \).

From the orders of \( x, y, \) and \( z \) it results the corresponding orders between the preferences \( C_1, C_2, \) and \( C_3 \).

**Example 3.**

Let’s suppose that the sources are not equally reliable

First source is five times less reliable than the second, while the third source is twice more reliable that the second one. Then the parameterized system:

\[
\begin{align*}
x &= \alpha_1 y^2  \\
y &= 2\alpha_2 z  \\
z^2 &= 3\alpha_3 x
\end{align*}
\]

since \( \alpha_1 = 3\alpha_2 \) which means that we need to discount the first equation three times more than the second, and \( \alpha_2 = \frac{\alpha_3}{4} \) which means that we need to discount the third equation a quarter of the second equation’s discount.

Denote \( \alpha_3 = \alpha \), then:

\[
\begin{align*}
x &= 3\alpha y^2  \\
y &= 2az  \\
z^2 &= \frac{3\alpha}{4} x
\end{align*}
\]

whence

\[
x = 3\alpha (2az)^2 = 12\alpha^3 z^2
\]

and

\[
x = \frac{4}{3\alpha} z^2
\]

therefore

\[
12\alpha^3 z^2 = \frac{4}{3\alpha} z^2, \text{ or } 36\alpha^4 = 4
\]

Thus

\[
\alpha = \frac{1}{\sqrt[4]{9}} = 0.485.
\]

The algebraic general solution is:

\[
\langle 12\alpha^3 z^2, 2az, z \rangle = \langle 1.557z^2, 0.970z, z \rangle = \begin{pmatrix} 1.557z \\ 1.557z + 1.854 \end{pmatrix}.
\]

And in a similar way, as we did for Example 2, we may discuss upon parameter \( z > 0 \) the order of preferences \( C_1, C_2, \) and \( C_3 \).
Reference: