In this article one builds a class of recursive sets, one establishes properties of these sets and one proposes applications. This article widens some results of [1].

1) **Definitions, properties.**

One calls recursive sets the sets of elements which are built in a recursive manner: let $T$ be a set of elements and $f_i$ for $i$ between 1 and $s$, of operations $n_i$, such that $f_i : T^{n_i} \rightarrow T$. Let’s build by recurrence the set $M$ included in $T$ and such that:

**(Def. 1)** 1°) certain elements $a_1,...,a_n$ of $T$, belong to $M$.

2°) if $(\alpha_{i_1},...,\alpha_{i_s})$ belong to $M$, then $f_i(\alpha_{i_1},...,\alpha_{i_s})$ belong to $M$ for all $i \in \{1,2,...,s\}$.

3°) each element of $M$ is obtained by applying a number finite of times the rules 1° or 2°.

We will prove several properties of these sets $M$, which will result from the manner in which they were defined. The set $M$ is the representative of a class of recursive sets because in the rules 1° and 2°, by particularizing the elements $a_1,...,a_n$ respectively $f_1,...,f_s$ one obtains different sets.

**Remark 1:** To obtain an element of $M$, it is necessary to apply initially the rule 1.

**(Def. 2)** The elements of $M$ are called elements $M$-recursive.

**(Def. 3)** One calls order of an element $a$ of $M$ the smallest natural $p \geq 1$ which has the property that $a$ is obtained by applying $p$ times the rule 1° or 2°.

One notes $M_p$ the set which contains all the elements of order $p$ of $M$. It is obvious that $M_1 = \{a_1,...,a_n\}$.

$$M_2 = \bigcup_{i=1}^s \left( \bigcup_{(\alpha_{i_1},...,\alpha_{i_s}) \in M_1^{n_i}} f_i(\alpha_{i_1},...,\alpha_{i_s}) \right) \setminus M_1.$$ 

One withdraws $M_1$ because it is possible that $f_j(a_{j_1},...,a_{j_m}) = a_i$ which belongs to $M_1$, and thus does not belong to $M_2$.

One proves that for $k \geq 1$ one has:
\[ M_{k+1} = \bigcup_{i=1}^{s} \left( \bigcup_{(\alpha_1, \ldots, \alpha_n) \in \prod_i} f_i(\alpha_1, \ldots, \alpha_n) \right) \setminus \bigcup_{h=1}^{k} M_h \]

where each

\[ \prod^{(i)}_k = \left\{ (\alpha_1, \ldots, \alpha_n) / \alpha_{j_h} \in M_{q_j}, j \in \{1, 2, \ldots, n_i\}; 1 \leq q_j \leq k \text{ and at least an element } \alpha_{j_h} \in M_k, 1 \leq j_h \leq n_i \right\}. \]

The sets \( M_p, \ p \in \mathbb{N}^\ast \) form a partition of the set \( M \).

**Theorem 1:**

\[ M = \bigcup_{p \in \mathbb{N}^\ast} M_p, \text{ where } \mathbb{N}^\ast = \{1, 2, 3, \ldots\}. \]

**Proof:**

From the rule 1° it results that \( M_1 \subseteq M \).

One supposes that this propriety is true for values which are less than \( p \). It results that \( M_p \subseteq M \), because \( M_p \) is obtained by applying the rule 2° to the elements of \( \bigcup_{i=1}^{p-1} M_i \).

Thus \( \bigcup_{p \in \mathbb{N}^\ast} M_p \subseteq M \). Reciprocally, one has the inclusion in the contrary sense in accordance with the rule 3°.

**Theorem 2:** The set \( M \) is the smallest set, which has the properties 1° and 2°.

**Proof:**

Let \( R \) be the smallest set having properties 1° and 2°. One will prove that this set is unique.

Let’s suppose that there exists another set \( R' \) having properties 1° and 2°, which is the smallest. Because \( R \) is the smallest set having these proprieties, and because \( R' \) has these properties also, it results that \( R \subseteq R' \); of an analogue manner, we have \( R' \subseteq R \): therefore \( R = R' \).

It is evident that \( M' \subseteq R \). One supposes that \( M_i \subseteq R \) for \( 1 \leq i < p \). Then (rule 3°), and taking in consideration the fact that each element of \( M_p \) is obtained by applying rule 2° to certain elements of \( M_i \), \( 1 \leq i < p \) it results that \( M_p \subseteq R \). Therefore \( \bigcup_{p \in \mathbb{N}^\ast} M_p \subseteq R \) \( (p \in \mathbb{N}^\ast) \), thus \( M \subseteq R \). And because \( R \) is unique, \( M = R \).

**Remark 2.** The theorem 2 replaces the rule 3° of the recursive definition of the set \( M \) by: “ \( M \) is the smallest set that satisfies proprieties 1° and 2°°°.

**Theorem 3:** \( M \) is the intersection of all the sets of \( T \) which satisfy conditions 1° and 2°.

**Proof:**
Let \( T_{12} \) be the family of all sets of \( T \) satisfying the conditions \( 1^o \) and \( 2^o \). We note \( I = \bigcap_{A \in T_{12}} A \).

\( I \) has the properties \( 1^o \) and \( 2^o \) because:
1) For all \( i \in \{1,2,\ldots,n\} \), \( a_i \in I \), because \( a_i \in A \) for all \( A \) of \( T_{12} \).
2) If \( \alpha_{i_1},\ldots,\alpha_{i_m} \in I \), it results that \( \alpha_{i_1},\ldots,\alpha_{i_m} \) belong to \( A \) that is \( A \) of \( T_{12} \).

Therefore,
\[
\forall i \in \{1,2,\ldots,s\}, \quad f_i(\alpha_{i_1},\ldots,\alpha_{i_m}) \in A \quad \text{which is} \quad A \quad \text{of} \quad T_{12}, \quad \text{therefore} \quad f_i(\alpha_{i_1},\ldots,\alpha_{i_m}) \in I
\]
for all \( i \) from \( \{1,2,\ldots,s\} \).

From theorem 2 it results that \( M \subseteq I \).

Because \( M \) satisfies the conditions \( 1^o \) and \( 2^o \), it results that \( M \in T_{12} \), from which \( I \subseteq M \). Therefore \( M = I \).

(Def. 4) A set \( A \subseteq I \) is called closed for the operation \( f_i \) if and only if for all \( \alpha_{i_1},\ldots,\alpha_{i_m} \) of \( A \), one has \( f_i(\alpha_{i_1},\ldots,\alpha_{i_m}) \) belong to \( A \).

(Def. 5) A set \( A \subseteq T \) is called closed \( M \)-recursive if and only if:
1) \( \{a_1,\ldots,a_n\} \subseteq A \).
2) \( A \) is closed in respect to operations \( f_1,\ldots,f_s \).

With these definitions, the precedent theorems become:

**Theorem 2'**: the set \( M \) is the smallest closed \( M \)-recursive set.

**Theorem 3'**: \( M \) is the intersection of all closed \( M \)-recursive sets.

(Def. 6) The system of elements \( \langle \alpha_{i_1},\ldots,\alpha_{i_m} \rangle \), \( m \geq 1 \) and \( \alpha_i \in T \) for \( i \in \{1,2,\ldots,m\} \), constitute a description \( M \)-recursive for the element \( \alpha \), if \( \alpha_m = \alpha \) and that each \( \alpha_i \) (\( i \in \{1,2,\ldots,m\} \)) satisfies at least one of the proprieties:
1) \( \alpha_i \in \{a_1,\ldots,a_n\} \).
2) \( \alpha_i \) is obtained starting with the elements which precede it in the system by applying the functions \( f_j \), \( 1 \leq j \leq s \) defined by property \( 2^o \) of (Def. 1).

(Def. 7) The number \( m \) of this system is called the length of the \( M \)-recursive description for the element \( \alpha \).

**Remark 3**: If the element \( \alpha \) admits a \( M \)-recursive description, then it admits an infinity of such descriptions.

Indeed, if \( \langle \alpha_{i_1},\ldots,\alpha_{i_m} \rangle \) is a \( M \)-recursive description of \( \alpha \) then
\[
\left\{ a_1,\ldots,a_1,\alpha_1,\ldots,\alpha_m \right\}_{h \text{ times}}
\]
is also a \( M \)-recursive description for \( \alpha \), \( h \) being able to take all values from \( \infty \).
**Theorem 4:** The set $M$ is identical with the set of all elements of $T$ which admit a $M$-recursive description.

*Proof:* let $D$ be the set of all elements, which admit a $M$-recursive description. We will prove by recurrence that $M_p \subseteq D$ for all $p$ of $\infty^\ast$.

For $p = 1$ we have: $M_1 = \{a_1, \ldots, a_n\}$, and the $a_j$, $1 \leq j \leq n$ having as $M$-recursive description: $< a_j >$. Thus $M_1 \subseteq D$. Let’s suppose that the property is true for the values smaller than $p$. $M_p$ is obtained by applying the rule $2^n$ to the elements of $\bigcup_{i=1}^{p-1} M_i$; $\alpha \in M_p$ implies that $\alpha \in f_j(\alpha_{i1}, \ldots, \alpha_{in})$ and $\alpha_i \in M_{h_j}$ for $h_j < p$ and $1 \leq j \leq n_i$. But $a_j$, $1 \leq j \leq n_i$, admits $M$-recursive descriptions according to the hypothesis of recurrence, let’s have $\langle \beta_{j1}, \ldots, \beta_{jn} \rangle$. Then $\langle \beta_{j1}, \ldots, \beta_{jn}, \beta_{2j1}, \ldots, \beta_{2jn}, \ldots, \beta_{njn}, \alpha \rangle$ constitute a $M$-recursive description for the element $\alpha$. Therefore if $\alpha$ belongs to $D$, then $M_p \subseteq D$ which is $M = \bigcup_{p=\infty} M_p \subseteq D$.

Reciprocally, let $x$ belong to $D$. It admits a $M$-recursive description $\langle \beta_1, \ldots, \beta_t \rangle$ with $\beta_t = x$. It results by recurrence by the length of the $M$-recursive description of the element $x$, that $x \in M$. For $t = 1$ we have $\langle \beta_1 \rangle$, $\beta_1 = x$ and $\beta_1 \in \{a_1, \ldots, a_n\} \subseteq M$. One supposes that all elements $y$ of $D$ which admit a $M$-recursive description of a length inferior to $t$ belong to $M$. Let $x \in D$ be described by a system of length $t$: $\langle \beta_1, \ldots, \beta_t \rangle$, $\beta_t = x$. Then $x \in \{a_1, \ldots, a_n\} \subseteq M$, where $x$ is obtained by applying the rule $2^n$ to the elements which precede it in the system: $\beta_1, \ldots, \beta_{t-1}$. But these elements admit the $M$-recursive descriptions of length which is smaller that $t$: $\langle \beta_1 \rangle, \langle \beta_1, \beta_2 \rangle, \ldots, \langle \beta_1, \ldots, \beta_{t-1} \rangle$. According to the hypothesis of the recurrence, $\beta_1, \ldots, \beta_{t-1}$ belong to $M$. Therefore $\beta_t$ belongs also to $M$. It results that $M \equiv D$.

**Theorem 5:** Let $b_1, \ldots, b_q$ be elements of $T$, which are obtained from the elements $a_1, \ldots, a_n$ by applying a finite number of times the operations $f_1, f_2, \ldots$, or $f_s$. Then $M$ can be defined recursively in the following mode:

1) Certain elements $a_1, \ldots, a_n, b_1, \ldots, b_q$ of $T$ belong to $M$.

2) $M$ is closed for the applications $f_i$, with $i \in \{1, 2, \ldots, s\}$.

3) Each element of $M$ is obtained by applying a finite number of times the rules (1) or (2) which precede.

*Proof:* evident. Because $b_1, \ldots, b_q$ belong to $T$, and are obtained starting with the elements $a_1, \ldots, a_n$ of $M$ by applying a finite number of times the operations $f_i$, it results that $b_1, \ldots, b_q$ belong to $M$. 

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Theorem 6: Let’s have \( g_j, \ 1 \leq j \leq r \), of the operations \( n_j \), where \( g_j : T^{n_j} \rightarrow T \) such that \( M \) to be closed in rapport to these operations. Then \( M \) can be recursively defined in the following manner:

1) Certain elements \( a_1,...,a_n \) de \( T \) belong to \( M \).
2) \( M \) is closed for the operations \( f_i, \ i \in \{1,2,...,s\} \) and \( g_j, \ j \in \{1,2,...,r\} \).
3) Each element of \( M \) is obtained by applying a finite number of times the precedent rules.

Proof is simple: Because \( M \) is closed for the operations \( g_j \) (with \( j \in \{1,2,...,r\} \)), one has, that for any \( \alpha_{j_1},...,\alpha_{j_n} \) from \( M \), \( g_j(\alpha_{j_1},...,\alpha_{j_n}) \in M \) for all \( j \in \{1,2,...,r\} \).

From the theorems 5 and 6 it results:

Theorem 7: The set \( M \) can be recursively defined in the following manner:

1) Certain elements \( a_1,...,a_n,b_1,...,b_q \) of \( T \) belong to \( M \).
2) \( M \) is closed for the operations \( f_i, \ i \in \{1,2,...,s\} \) and for the operations \( g_j, \ j \in \{1,2,...,r\} \) previously defined.
3) Each element of \( M \) is defined by applying a finite number of times the previous 2 rules.

(Def. 8) The operation \( f_i \) conserves the property \( P \) iff for any elements \( \alpha_{i_1},...,\alpha_{i_n} \) having the property \( P \), \( f_i(\alpha_{i_1},...,\alpha_{i_n}) \) has the property \( P \).

Theorem 8: If \( a_1,...,a_n \) have the property \( P \), and if the functions \( f_1,...,f_s \) preserve this property, then all elements of \( M \) have the property \( P \).

Proof:

\[ M = \bigcup_{p=1}^{\infty} M_p. \] The elements of \( M_i \) have the property \( P \).

Let’s suppose that the elements of \( M_i \) for \( i < p \) have the property \( P \). Then the elements of \( M_p \) also have this property because \( M_p \) is obtained by applying the operations \( f_1,f_2,...,f_s \) to the elements of: \( \bigcup_{i=1}^{p} M_i \), elements which have the property \( P \).

Therefore, for any \( p \) of \( \infty \), the elements of \( M_p \) have the property \( P \).

Thus all elements of \( M \) have it.

Corollary 1: Let’s have the property \( P: "x \) can be represented in the form \( F(x)" \).

If \( a_1,...,a_n \) can be represented in the form \( F(a_1),...,F(a_n) \), and if \( f_1,...,f_s \) maintains the property \( P \), then all elements \( \alpha \) of \( M \) can be represented in the form \( F(\alpha) \).

Remark. One can find more other equivalent def. of \( M \).
2) APPLICATIONS, EXAMPLES.

In applications, certain general notions like: \( M \) - recursive element, \( M \) - recursive description, \( M \) - recursive closed set will be replaced by the attributes which characterize the set \( M \). For example in the theory of recursive functions, one finds notions like: recursive primitive functions, primitive recursive description, primitively recursive closed sets. In this case "\( M \)" has been replaced by the attribute "primitive" which characterizes this class of functions, but it can be replaced by the attributes "general", "partial".

By particularizing the rules 1° and 2° of the def. 1, one obtains several interesting sets:

**Example 1:** (see [2], pp. 120-122, problem 7.97).

**Example 2:** The set of terms of a sequence defined by a recurring relation constitutes a recursive set.

Let's consider the sequence: \( a_{n+k} = f(a_n, a_{n+1}, \ldots, a_{n+k-1}) \) for all \( n \) of \( \infty^* \), with \( a_i = a_0^i, \ 1 \leq i \leq k \). One will recursively construct the set \( A = \{a_m\}_{m \in \infty} \) and one will define in the same time the position of an element in the set \( A \):

1°) \( a_1^0, \ldots, a_k^0 \) belong to \( A \), and each \( a_i^0 \ (1 \leq i \leq k) \) occupies the position \( i \) in the set \( A \);

2°) if \( a_n, a_{n+1}, \ldots, a_{n+k-1} \) belong to \( A \), and each \( a_j \) for \( n \leq j \leq n+k-1 \) occupies the position \( j \) in the set \( A \), then \( f(a_n, a_{n+1}, \ldots, a_{n+k-1}) \) belongs to \( A \) and occupies the position \( n+k \) in the set \( A \).

3°) each element of \( B \) is obtained by applying a finite number of times the rules 1° or 2°.

**Example 3:** Let \( G = \{e, a_1^0, a_2^0, \ldots, a^n\} \) be a cyclic group generated by the element \( a \). Then \( G \) can be recursively defined in the following manner:

1°) \( a \) belongs to \( G \).

2°) if \( b \) and \( c \) belong to \( G \) then \( b \cdot c \) belongs to \( G \).

3°) each element of \( G \) is obtained by applying a finite number of times the rules 1 or 2.

**Example 4:** Each finite set \( ML = \{x_1, x_2, \ldots, x_n\} \) can be recursively defined (with \( ML \subseteq T \)):

1°) The elements \( x_1, x_2, \ldots, x_n \) of \( T \) belong to \( ML \).

2°) If \( a \) belongs to \( ML \), then \( f(a) \) belongs to \( ML \), where \( f : T \rightarrow T \) such that \( f(x) = x \);

3°) Each element of \( ML \) is obtained by applying a finite number of times the rules 1° or 2°.

**Example 5:** Let \( L \) be a vectorial space on the commutative corps \( K \) and \( \{x_1, \ldots, x_m\} \) be a base of \( L \). Then \( L \) can be recursively defined in the following manner:

1°) \( x_1, \ldots, x_m \) belong to \( L \);

2°) if \( x, y \) belong to \( L \) and if \( a \) belongs to \( K \), then \( x \perp y \ y \) belong to \( L \) and \( a \cdot x \) belongs to \( L \);
3°) each element of \( L \) is recursively obtained by applying a finite number of times the rules 1° or 2°.

(The operators \( \perp \) and \( * \) are respectively the internal and external operators of the vectorial space \( L \).)

**Example 6:** Let \( X \) be an \( A \)-module, and \( M \subset X \) \((M \neq \emptyset)\), with \( M = \{ x_i \}_{i \in I} \).

The sub-module generated by \( M \) is:

\[
\langle M \rangle = \{ x \in X / x = a_1 x_1 + \ldots + a_n x_n, \ a_i \in A, \ x_i \in M, \ i \in 1,\ldots,n \}
\]

can recursively defined in the following way:

1°) for all \( i \) of \( \{1,2,\ldots,n\} \), \( 1,2,\ldots,n \perp x_i \in \langle M \rangle \);  
2°) if \( x \) and \( y \) belong to \( \langle M \rangle \) and \( a \) belongs to \( A \), then \( x + y \) belongs to \( \langle M \rangle \), and \( ax \) also;  
3°) each element of \( \langle M \rangle \) is obtained by applying a finite number of times the rules 1° or 2°.

In accordance to the paragraph 1 of this article, \( \langle M \rangle \) is the smallest sub-set of \( X \) that verifies the conditions 1° and 2°, that is \( \langle M \rangle \) is the smallest sub-module of \( X \) that includes \( M \). \( \langle M \rangle \) is also the intersection of all the subsets of \( X \) that verify the conditions 1° and 2°, that is \( \langle M \rangle \) is the intersection of all sub-modules of \( X \) that contain \( M \). One also directly refines some classic results from algebra.

One can also talk about sub-groups or ideal generated by a set: one also obtains some important applications in algebra.

**Example 7:** One also obtains like an application the theory of formal languages, because, like it was mentioned, each regular language (linear at right) is a regular set and reciprocally. But a regular set on an alphabet \( \Sigma = \{ a_1,\ldots,a_n \} \) can be recursively defined in the following way:

1°) \( \emptyset, \{ \varepsilon \}, \{ a_1 \},\ldots,\{ a_n \} \) belong to \( R \).  
2°) if \( P \) and \( Q \) belong to \( R \), then \( P \cup Q, PQ, \) and \( P^* \) belong to \( R \), with

\[
P \cup Q = \{ x \in P \text{ or } x \in Q \}, \quad PQ = \{ xy \mid x \in P \text{ and } y \in Q \}, \quad \text{and} \quad P^* = \bigcup_{n=0}^{\infty} P^n
\]

and, by convention, \( P^0 = \{ \varepsilon \} \).

3°) Nothing else belongs to \( R \) other that those which are obtained by using 1° or 2°.

From which many properties of this class of languages with applications to the programming languages will result.

**REFERENCES:**