The Dual of a Theorem relative to the Orthocenter of a Triangle

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In [1] we introduced the notion of Bobillier transversal relative to a point $O$ in the plane of a triangle $ABC$; we use this notion in what follows.

We transform by duality with respect to a circle $C$ $o, r$ the following theorem relative to the orthocenter of a triangle.

**Theorem 1.** If $ABC$ is a nonisosceles triangle, $H$ its orthocenter, and $AA_1, BB_1, CC_1$ are cevians of a triangle concurrent at point $Q$ different from $H$, and $M, N, P$ are the intersections of the perpendiculars taken from $H$ on given cevians respectively, with $BC, CA, AB$, then the points $M, N, P$ are collinear.

**Proof.** We note with $\alpha = m \triangleleft BAA_1$; $\beta = m \triangleleft CBB_1$; $\gamma = m \triangleleft ACC_1$, see Figure 1. According to Ceva’s theorem, trigonometric form, we have the relation:

$$\frac{\sin \alpha}{\sin A - \alpha} \cdot \frac{\sin \beta}{\sin B - \beta} \cdot \frac{\sin \gamma}{\sin C - \gamma} = 1. \quad (1)$$

We notice that:

$$\frac{MB}{MC} = \frac{\text{Arie } MHB}{\text{Arie}(MHC)} = \frac{MHB \cdot \sin sin MHB}{MH \cdot HC \cdot \sin MHC}.$$

Because: $\triangleleft MHB \equiv \triangleleft A_1AC$ as angles of perpendicular sides, it follows that

$$m \triangleleft MHB = m A - \alpha.$$
Therewith \( m \angle M H C = m \angle M H B + m \angle B H C = 180^\circ \alpha \).

We thus get that:

\[
\frac{MB}{MC} = \frac{\sin A - \alpha}{\sin \alpha} \cdot \frac{HB}{HC}.
\]

Analogously, we find that:

\[
\frac{NC}{NA} = \frac{\sin B - \beta}{\sin \beta} \cdot \frac{HC}{HA}.
\]

\[
\frac{PA}{PB} = \frac{\sin C - \gamma}{\sin \gamma} \cdot \frac{HA}{HB}.
\]

Applying the reciprocal of Menelaus' theorem, we find, in view of (1), that:

\[
\frac{MB}{MC} \cdot \frac{HC}{HA} \cdot \frac{PA}{PB} = 1.
\]

This shows that \( M, N, P \) are collinear.

**Note.** Theorem 1 is true even if \( ABC \) is an obtuse, nonisosceles triangle. The proof is adapted analogously.

**Theorem 2 (The Dual of the Theorem 1).** If \( ABC \) is a triangle, \( O \) a certain point in his plan, and \( A_1, B_1, C_1 \) Bobillier transversals relative to \( O \) of \( ABC \) triangle, as well as \( A_2 - B_2 - C_2 \) a certain transversal in \( ABC \), and the perpendiculars in \( O \), and on \( OA_2, OB_2, OC_2 \) respectively, intersect the Bobillier
transversals in the points $A_3, B_3, C_3$, then the cevians $AA_3, BB_3, CC_3$ are concurrent.

**Proof.** We convert by duality with respect to a circle $\mathcal{C} \circ o, r$ the figure indicated by the statement of this theorem, i.e. Figure 2. Let $a, b, c$ be the polars of the points $A, B, C$ with respect to the circle $\mathcal{C} \circ o, r$. To the lines $BC, CA, AB$ will correspond their poles $A'4 = bnc; \ B'4 = cna; \ C'4 = anb$.

To the points $A_1, B_1, C_1$ will respectively correspond their polars $a_1, b_1, c_1$ concurrent in transversal’s pole $A_1 - B_1 - C_1$.

Since $OA_1 \perp OA$, it means that the polars $a$ and $a_1$ are perpendicular, so $a_1 \perp B'C'$, but $a_1$ pass through $A'$, which means that $Q'$ contains the height from $A'$ of $A'B'C'$ triangle and similarly $b_1$ contains the height from $B'$ and $c_1$ contains the height from $C'$ of $A'B'C'$ triangle. Consequently, the pole of $A_1 - B_1 - C_1$ transversal is the orthocenter $H'$ of $A'B'C'$ triangle. In the same way, to the points $A_2, B_2, C_2$ will correspond the polars to $a_2, b_2, c_2$ which pass respectively through $A', B', C'$ and are concurrent in a point $Q'$, the pole of the line $A_2 - B_2 - C_2$ with respect to the circle $\mathcal{C} \circ o, r$. Given $OA_2 \perp OA_3$, it means that the polars $a_2$ and $a_3$ are perpendicular, $a_2$ correspond to the cevian $A'Q'$, also $a_3$ passes through the the pole of the transversal $A_1 - B_1 - C_1$, so through
$H'$, in other words $Q_3$ is perpendicular taken from $H'$ on $A'Q'$; similarly, $b_2 \perp b_3, c_2 \perp c_3$, so $b_3$ is perpendicular taken from $H'$ on $C'Q'$. To the cevian $AA_3$ will correspond by duality considered to its pole, which is the intersection of the polars of $A$ and $A_3$, i.e. the intersection of lines $a$ and $a_3$, namely the intersection of $B'C'$ with the perpendicular taken from $H'$ on $A'Q'$; we denote this point by $M'$. Analogously, we get the points $N'$ and $P'$. Thereby, we got the configuration from Theorem 1 and Figure 1, written for triangle $A'B'C'$ of orthocenter $H'$. Since from Theorem 1 we know that $M', N', P'$ are collinear, we get the the cevians $AA_3, BB_3, CC_3$ are concurrent in the pole of transversal $M' - N' - P'$ with respect to the circle $C_0, r$, and Theorem 2 is proved.

References

[1] Ion Patrascu, Florentin Smarandache: „The Dual Theorem concerning Aubert Line“.