# THE DUAL THEOREM RELATIVE TO THE SIMSON'S LINE

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**Abstract**

In this article we elementarily prove some theorems on the poles and polars theory, we present the transformation using duality and we apply this transformation to obtain the dual theorem relative to the Samson's line.

## I. POLE AND POLAR IN RAPPORT TO A CIRCLE

**Definition 1.** Considering the circle $C(O,R)$, the point $P$ in its plane $P \neq O$ and the point $P'$ such that $\overrightarrow{OP} \cdot \overrightarrow{OP'} = R^2$. It is said about the perpendicular $p$ constructed in the point $P'$ on the line $OP$ that it is the point's $P$ **polar**, and about the point $P$ that it is the line's $p$ **pole**.

**Observations**

1. If the point $P$ belongs to the circle, its polar is the tangent in $P$ to the circle $C(O,R)$.

Indeed, the relation $\overrightarrow{OP} \cdot \overrightarrow{OP'} = R^2$ gives that $P' = P$.

2. If $P$ is interior to the circle, its polar $p$ is a line exterior to the circle.

3. If $P$ and $Q$ are two points such that $m(\widehat{POQ}) = 90^\circ$, and $p,q$ are their polars, from the definition results that $p \perp q$.

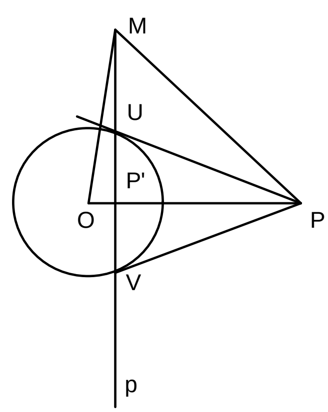


**Fig. 1**

**Proposition 1.**

If the point $P$ is external to the circle $C(O,R)$, its polar is determined by the contact points with the circle of the tangents constructed from $P$ to the circle

**Proof**

Let $U$ and $V$ be the contact points of the tangents constructed from $P$ to the circle $C(O,R)$ (see fig.1). In the right triangle $OUP$, if $P''$ is the orthogonal projection of $U$ on $OP$, we have $OU^2 = OP''\cdot OP$ (the cathetus theorem), but also $OP'\cdot OP = R^2$, it results that $P'' = P'$ and therefore $U$ belongs to the polar of $P$. Similarly $V$ belongs to the polar, therefore $UV$ is the polar of $P$.

**Observation**

From the Proposition 1 it results the construction's method of the polar of an exterior point to a circle.

**Theorem 1. (The Polar Characterization)**

The point $M$ belongs to the polar of the point $P$ in rapport to the circle $C(O,R)$ if and only if

$$MO^2 - MP^2 = 2R^2 - OP^2.$$

**Proof**

If $M$ is an arbitrary point on the polar of the point $P$ in rapport to the circle $C(O,R)$, then

$$MP' \perp OP$$

(see fig. 1) and

$$MO^2 - MP^2 = \left(P'O^2 + P'M^2\right) - \left(P'P^2 + P'M^2\right) = P'O^2 -$$

$$-P'P^2 = OU^2 - P'U^2 + P'U^2 - PU^2 = R^2 - \left(OP^2 - R^2\right) = 2R^2 - OP^2.$$

Reciprocally, if $M$ is in the circle's plane such that

$$MO^2 - MP^2 = 2R^2 - OP^2.$$

We denote with $M'$ the projection of $M$ on $OP$, and we have

$$M'O^2 - M'P^2 = \left(MO^2 - M'M^2\right) - \left(MP^2 - \ -M'M^2\right) = MO^2 - MP^2 = 2R^2 - OP^2.$$

On the other side

$$P'O^2 - P'P^2 = 2R^2 - OP^2.$$

From

$$M'O^2 - M'P^2 = P'O^2 - P'P^2$$

it results that

$$M' = P',$$

therefore $M$ belongs to the polar of the point $P$.

**Theorem 2. (Philippe de la Hire)**

If $P, Q, R$ are points that don't belong to the circle $C(O,R)$ and $p, q, r$ are their polars in rapport to the circle, then

$1^\circ$ $P \in q \Leftrightarrow Q \in p$ (If a point belongs to the polar of another point, then also a second point belongs to the polar of the first point in rapport to a circle)

$2^{\circ}$ $r = PQ \Leftrightarrow R \in p \cap q$ (The pole of a line that passes through two points is the intersection of the polars of the two points).

**Proof:**
$1^{\circ}$ From the theorem 1 we have

$$P \in q \Leftrightarrow PO^2 - PQ^2 = 2R^2 - OQ^2 .$$

Then

$$QO^2 - OP^2 = 2R^2 - OP^2 \Leftrightarrow Q \in p .$$

$2^{\circ}$ Let's consider $R \in p \cap q$; from $1^{\circ}$ results $P \in r$ and $Q \in r$ therefore

$$r = PQ .$$

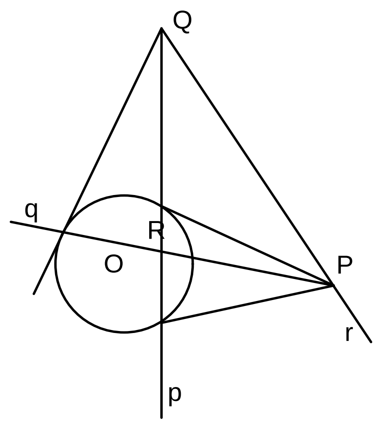


**Fig. 2**

**Observations**
1. From the theorem 2 we retain:
The polar of a point which is the intersection of two given lines is the line determined by the poles of those lines.
2. The poles of some concurrent lines are collinear and reciprocally, the polars of some collinear points are concurrent.

## II. THE TRANSFORMATION THROUGH DUALITY

The duality in rapport with a circle $C(O, R)$ is the geometric transformation that associates to any point $P \neq O$ its polar, and which associates to a line from plane its pole.

Through duality, practically, the role of the lines and of the points are permutated, such that to a figure $F$ formed of points and lines, through duality it corresponds to it a new figure $F'$ formed from lines (the figure's $F$ polars) and of points (the poles of the figure's $F$ lines) in rapport to a given circle.

The duality was introduced by the French mathematician Victor Poncelet in 1822.

When the figure $F$ is formed of points, lines and eventually a circle, transforming it through duality in rapport with the circle we will still be in the elementary geometry domain and we obtain a new figure $F'$, its elements and properties being duals to those of the figure $F$.

From the proved theorems we retain:

- If a point is located on a line, through its duality it will correspond its polar that passes through the line's pole in rapport to the circle.

- To the line determined by two points it corresponds, through duality in rapport with a circle, the intersection point of the polars of those two points.

- To the intersection's point of two lines it corresponds, through duality in rapport with a circle, the line determined by the poles of these lines.

## III. THE DUAL THEOREM RELATIVE TO THE SIMSON'S LINE

**Theorem 3. (The Simson's Line)**

If $A'B'C'$ is a triangle inscribed in the circle with the center in $O$ and $A_1', B_1', C_1'$ are the orthogonal projections of a point $M$ from the circle respectively on $B'C', C'A'$ and $A'B'$, then the points $A_1', B_1', C_1'$ are collinear.

We leave to the reader's attention the proof of this known theorem.

We transform through duality in rapport to the circumscribed circle to the triangle $A'B'C'$ the configuration of this theorem. To the points $A', B', C'$ correspond through duality their polars $a, b, c$, which are the tangents in $A', B', C'$ to the circle (see fig. 3), and to the point $M$ corresponds its polar $m$, the tangent in $M$ to circle.

To the line $A'B'$ it is associated through duality its pole $\{C\} = a \cap b$, similarly to the line $A'C'$ corresponds the point $\{B\} = a \cap c$ and to the line $B'C'$ corresponds $\{A\} = b \cap c$.

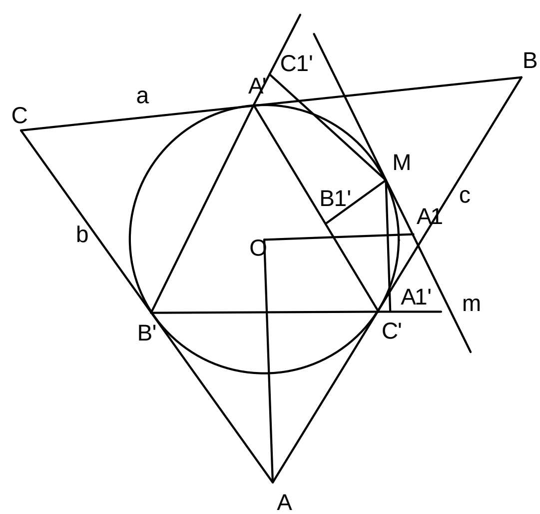


**Fig. 3**

Because $MA_1' \perp B'C'$ it results that their poles are situated on perpendicular lines that pass through $O$, therefore, if we denote with $A_1$ the line's pole $MA_1'$ we will find $A_1$ as the intersection of the perpendicular constructed in $O$ on $AO$ with the tangent $m$. Similarly we obtain the points $B_1$ and $C_1$.

Through the considered duality, to the point $A_1'$ corresponds its polar, which is $AA_1$ (because the pole of $MA_1'$ is $A_1$ and the pole of $B'C'$ is $A$).

Similarly to the point $B_1^{'}$ corresponds $BB_1$ and to the point $C_1^{'}$ through duality corresponds $CC_1$. Because the points $A_1^{'}, B_1^{'}, C_1^{'}$ are collinear (the Simson's line) it results that their polars $AA_1, BB_1, CC_1$ are concurrent in a point $S$ (the Simson's line's pole ).

We observe that the circumscribed circle to the triangle $A'B'C'$ becomes inscribed circle in the triangle $ABC$, and therefore we can formulate the following:

**Theorem 4 (The Dual Theorem of the Simson's Line).**

If $ABC$ is any triangle and $A_1, B_1, C_1$ are, respectively, the intersections of the perpendiculars constructed in the center $I$ of the inscribed circle in triangle on $AI, BI, CI$ with a tangent constructed to the inscribed circle, then the lines $AA_1, BB_1, CC_1$ are concurrent.

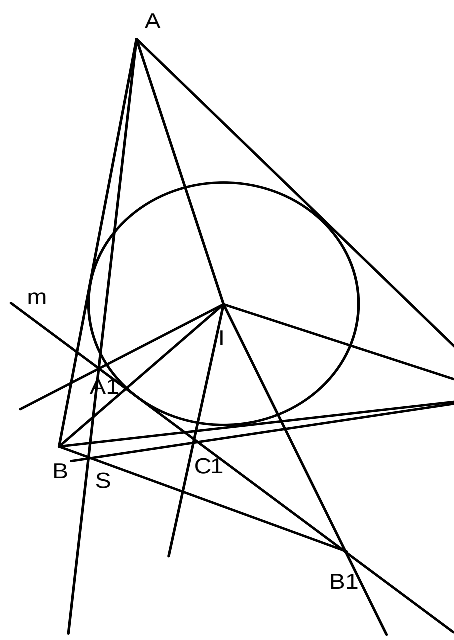


**Fig. 4**

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