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SMARANDACHE TYPE FUNCTIONS OBTAINED BY DUALITY

by

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Abstract

In this paper we extend the Smarandache function from the set N* of positive integers to the set Q of rationals.

Using the invertion formula this function is also regarded as a generating function.We make in evidence a procedure to construct (numerical) functions starting from a given function in two particular cases. Also some conections between the Riemann's zeta function are etablished.

1. Introduction

The Smarandache function [13] is a numerical function $S:N^* \to N$ defined by $S(n) = min \{ m! : m \text{ is divisible by } n \}$.

From the definition it results that if $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_t^{\alpha_t}$ (1)

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is the decomposition of n into primes then $S(n)=\max S(p_i^{\alpha_i})$ (2)

and moreover, if $[n_1,n_2]$ is the smallest common multiple of n_1 and n_2 then

$S([n_1,n_2])=\max\{S(n_1),S(n_2)\}$ (3)

The Smarandache function characterizes the prime numbers in the sense that a positive integer $p\geq 4$ is prime if end only if it is a fixed point of S.

From Legendre's formula:

$$m! = \prod_p p^{\sum_{i\geq 1}[\frac{m}{p^i}]} \quad (4)$$

it results [2] that if $a_n(p)=\frac{(p^n-1)}{(p-1)}$ and $b_n(p)=p^n$ then considering the standard numerical scale

$$[p] : b_0(p),b_1(p),...,b_n(p),...$$

and the generalized scale

$$[p] : a_0(p),a_1(p),...,a_n(p),...$$

we have

$$S(p^k)=p(\alpha_{[p]})_{(p)} \quad (5)$$

that is $S(p^k)$ is calculated multiplying by p the number obtenained writing the exponent α in the generalised scale [p] and "reading" it in the standard scale (p).
Let us observe that the calculus in the generalised scale [p] is essentilly different from the calculus in the usual scale (p), because the usual relationship $b_{n+1}(p)=pb_n(p)$ is modified in $a_{n+1}(p)=pa_n(p)+1$ (for more details see [2]).

In the following let us note $S_p(\alpha)=S(p^\alpha)$. In [3] it is proved that

$$S_p(\alpha)=(p-1)\alpha+\sigma_{[p]}(\alpha) \quad (6)$$

where $\sigma_{[p]}(\alpha)$ is the sum of the digits of α written in the scale [p] and also that

$$S_p(\alpha)=\frac{(p-1)^2}{p}(E_p(\alpha)+\alpha)+\frac{p-1}{p}\sigma_{(p)}(\alpha)+\sigma_{[p]}(\alpha) \quad (7)$$

where $\sigma_{(p)}(\alpha)$ is the sum of the digits of α written in the standard scale (p) and $E_p(\alpha)$ is the exponent of p in the decomposition into primes of α!. From (4) results that $E_p(\alpha)=\sum_{i\geq 1}[\frac{\alpha}{p^i}]$, where [h] is the integer part of h. It is also said [11] that

$$E_p(\alpha)=\frac{\alpha-\sigma_{(p)}(\alpha)}{p-1} \quad (8)$$

we can observe that this equality may be written as

$$E_p(\alpha)=([\frac{\alpha}{p}]_{(p)})_{[p]}$$

that is the exponent of p in the decomposition into primes of α! is obtained writing the integer part of α/p in the base (p) and reading in the scale [p].

Finally we note that in [1] it is proved that

$$S_p(\alpha)=p(\alpha-[\frac{\alpha}{p}]+[\frac{\sigma_{[p]}(\alpha)}{p}]) \quad (9)$$

From the definition of S it results that $S_p(E_p(\alpha))=p[\frac{\alpha}{p}]=\alpha-\alpha_p$ (α_p is the remainder of α with respect to the modulus m) and also that

$$E_p(S_p(\alpha))\geq\alpha; \quad E_p(S_p(\alpha)-1)<\alpha \quad (10)$$

so

$$\frac{S_p(\alpha)-\sigma_{(p)}(S_p(\alpha))}{p-1}\geq\alpha \ ; \ \frac{S_p(\alpha)-1-\sigma_{(p)}(S_p(\alpha)-1)}{p-1}<\alpha$$

Using (6) we obtain that $S_p(\alpha)$ is the unique solution of the system

$$\sigma_{(p)}(x)\leq\sigma_{[p]}(\alpha)\leq\sigma_{(p)}(x-1)+1 \quad (11),$$

2. Connection with cllasical numerical functions

It is said that Riemann's zeta function is $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$

We may etablishe a connection between the functions S_p and Riemann's function as follows:

Proposition 2.1. If $n=\prod_{i=1}^{t_n}p_i^{\alpha_{i_n}}$ is the decomposition into primes of the pozitive integer n then

$$\frac{\zeta(s-1)}{\zeta(s)}=\sum_{n=1}^{\infty}\prod_{i=1}^{t_n}\frac{S_{p_i}(p_i^{\alpha_{i_n}-1})-p_i}{p_i^{s\alpha_{i_n}}}$$

Proof. We first etablishe a connection with Euler's totient function φ. Let us observe that for $\alpha\geq 2$, $p^{\alpha-1}=(p-1)a_{\alpha-1}(p)+1$, so $\sigma_{[p]}(p^{\alpha-1})=p$. Then by means of (6) it results (for $\alpha\geq 2$) that

$$S_p(p^{\alpha-1})=(p-1)p^{\alpha-1}+\sigma_{[p]}(p^{\alpha-1})=\varphi(p^{\alpha})+p$$

Using the well known relation between φ and ζ given by

$$\frac{\zeta(s-1)}{\zeta(s)}=\sum_{n\geq 1}\frac{\varphi(n)}{n^n}$$

and (12) it results the required relation.

Let us remark also that if n is given by (1), then

$$\varphi(n)=\prod_{i=1}^{t}\varphi(p_i^{\alpha_i})=\prod_{i=1}^{t}(S_{p_i}(p_i^{\alpha_i-1})-p_i)$$

and

$$S(n)=\max(\varphi(p_i^{\alpha_i+1})+p_i)$$

Now it is said that $1+\varphi(p_i)+\ldots+\varphi(p_i^{\alpha_i})=p_i^{\alpha_i}$ and then

$$\sum_{k=1}^{\alpha_i-1}Sp_i(p_i^k)-(\alpha_i-1)p_i=p_i^{\alpha_i}$$

Consequently we may write

$$S(n)=\max(S\sum_{k=0}^{\alpha_i-1}Sp_i(p_i^k)-(\alpha_i-1)p_i)$$

To etablishe a connection with Mangolt's function let us note $\bigwedge=\min$, $\bigvee=\max$, $\bigwedge_d=$ the greatest common divisor and $\overset{d}{\bigvee}=$ the smallest common multiple.

We shall write also $n_1\bigwedge_d n_2=(n_1,n_2)$ and $n_1\overset{d}{\bigvee}n_2=[n_1,n_2]$.

The Smarandache function S may be regarded as a function from the lattice $\mathcal{L}_d=(N^*,\bigwedge_d,\overset{d}{\bigvee})$ into the lattice $\mathcal{L}=(N^*,\bigwedge,\bigvee)$ so that

$$S(\bigvee_{i=\overline{1,k}}n_i)=\bigvee_{i=\overline{1,k}}S(n_i)\quad(14)$$

Of course S is also order preserving in the sense that $n_1 \leq_d n_2 \Rightarrow S(n_1) < S(n_2)$

It is said [10] that if $(V, \wedge, \vee)$ is a finite lattice, $V = \{x_1, x_2, \ldots, x_n\}$ with the induced order $\leq$, then for every function $f: V \to N$ the asociated generating function is defined by

$$F(x) = \sum_{y \leq x} f(y) \quad (15)$$

Maglot's function Λ is

$$\Lambda(n) = \begin{cases} \ln p & if\ n = p^i \\ 0 & otherwise \end{cases}$$

The generating function of Λ in the lattice $\mathcal{L}_d$ is

$$F^d(n) = \sum_{k \leq_d n} \Lambda(k) = \ln n \quad (16)$$

The last equality follows from the fact that

$$k \leq_d n \Leftrightarrow k \wedge_d = k \Leftrightarrow k/n \quad \text{(k divides n)}$$

The generating function of Λ in the lattice $\mathcal{L}$ is the function Ψ

$$F(n) = \sum_{k \leq n} \Lambda(k) = \Psi_{(n)} = \ln[1,2,\ldots,n] \quad (17)$$

Then we have the diagram from below.
We observe that the definition of S is in a closed connection with the equalities (1.1) and (2.2) in this diagram. If we note the Mangolt's function by f then the relations

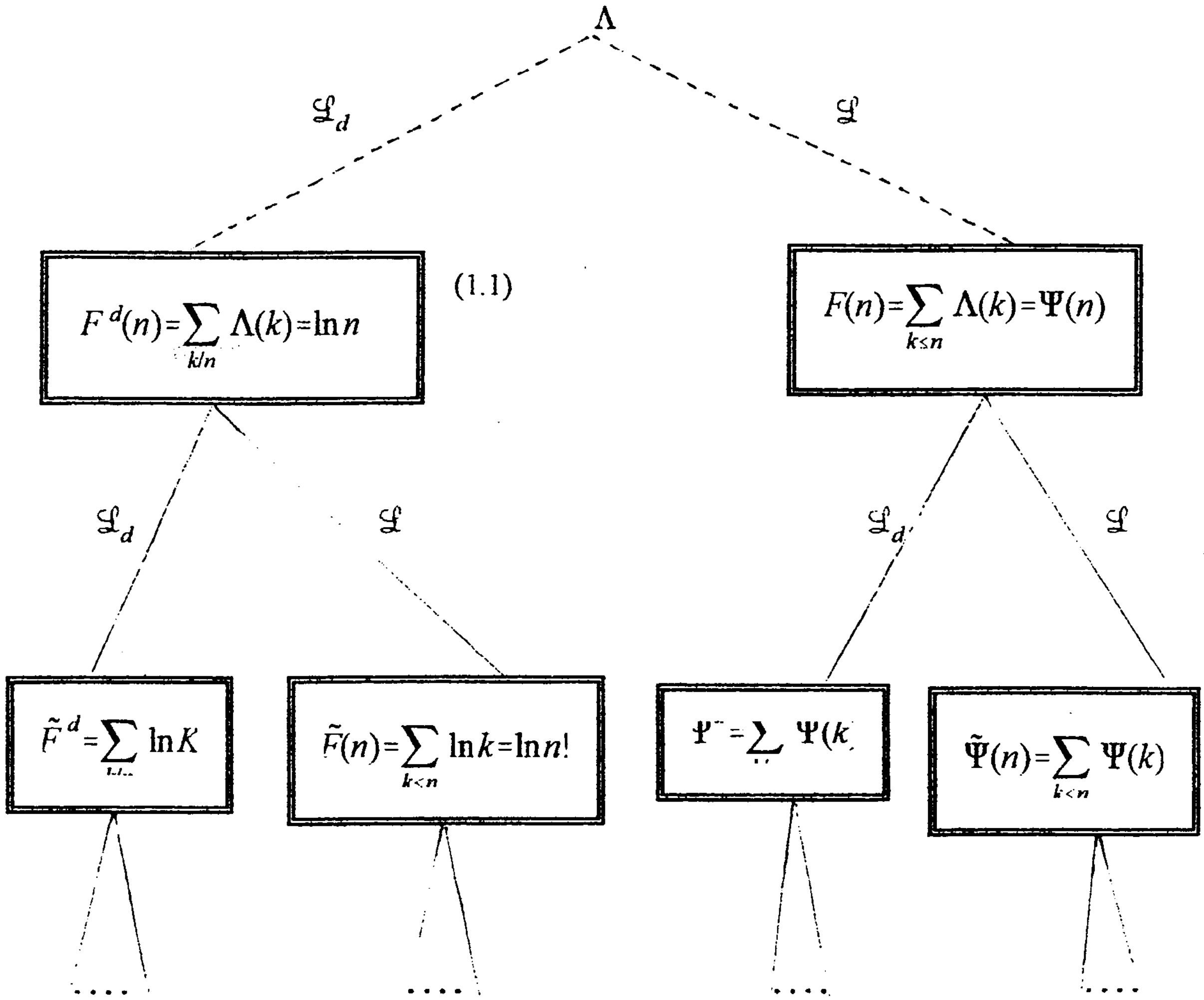
$$[1,2,\ldots,n] = e^{F(n)} = e^{f(1)} e^{f(2)} \ldots e^{f(n)} = e^{\Psi(n)}$$

$$n! = e^{\bar{F}} = e^{F^d(1)} e^{F^d(2)} \ldots e^{F^d(n)}$$

together with the definition of S suggest us to consider numerical functions of the form:

$$v(n)=\min\{m/n\leq_d[1,2,\ldots,m]\} \quad (18)$$

where will be detailed in the section 5.



3. The Smarandache function as generating function

Let V be a partial order set. A function $f:V\to N$ may be obtained from its generating function F defined as in (15), by the inversion formula

$$f(x)=\sum_{z\leq x}F(z)\mu(z,x) \quad (19)$$

where μ is Moebius function on V, that is $\mu : V\times V \to N$ satisfies:

$$(\mu_1)\ \mu(x,y)=0 \quad if\ x\leq y$$
$$(\mu_2)\ \mu(x,x)=1$$
$$(\mu_3)\ \sum_{x\leq y\leq z} \mu(x,y)=0\ \ if\ x<z$$

It is said [10] that if $V=\{1,2,...,n\}$ then for $(V,\leq_d)$ we have $\mu(x,y)=\mu(\frac{y}{x})$,

where $\mu(k)$ is the numerical Moebius function

$\mu(1)=1,\ \mu(k)=(-1)^t\ if\ k=p_1p_2...p_t\ and\ \mu(k)=0$ if k is divisible by the square of an integer d>1.

If f is the Smarandache function it results

$$F_s(n)=\sum_{d/n} S(n)$$

Until now it is not known a closed formula for F_S , but in [8] it is proved that

(I) $F_S(n)=n$ if and only if n is a prime , n=9, n=16 or n=24

(ii) $F_S(n)>n$ if and only if $n\in\{8,12,18,20\}$ or $n=2p$ with p a prime (hence it results $F_S(n)\leq n+4$ for every pozitive integer n) and in [2] it is showed that

(iii) $F(p_1p_2...p_t)=\sum_{i=1}^{t} 2^{i-1}p_i$

In this section we shall regard the Smarandache function as a a generating function that is using the invertion formula we shall construct the function s so that

$$s(n)=\sum_{d/n} \mu(d)S(\frac{n}{d}) \quad (20)$$

If n is given by (1) it results that

$$s(n)=\sum_{p_{i_1}p_{i_2}\cdots p_{i_r}}(-1)^r S\left(\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_r}}\right)$$

Let us consider $S(n)=\max S(p_i^{\alpha_i})=S(p_{i_0}^{\alpha_{i_0}})$. We distinguishe the following cases:

(a_1) if $S(p_{i_0}^{\alpha_{i_0}})\geq S(p_i^{\alpha_i})$ for all $i\neq i_0$ then we observe that the divisors d for whitch $\mu(d)\neq 0$ are of the form $d=1$ or $d=p_{i_1}p_{i_2}\cdots p_{i_r}$. A divisor of the last form may contais p_{i_0} or not, so using (2) it results

$$s(n)=S(p_{i_0}^{\alpha_{i_0}})(1-C_{t-1}^{1}+C_{t-1}^{2}+\ldots+(-1)^{t-1}C_{t-1}^{t-1})+$$
$$+S(p_{i_0}^{\alpha_{i_0}-1})(-1+C_{t-1}^{1}-C_{t-1}^{2}+\ldots+(-1)^{t}C_{t-1}^{t-1})$$

that is $s(n)=0$ if $t\geq 2$ or $S(p_{i_0}^{\alpha_{i_0}})=S(p_{i_0}^{\alpha_{i_0}-1})$ and $S(n)=p_{i_0}$ otherwise

(a_2) if there exist j_0 so that $S(p_{i_0}^{\alpha_{i_0}-1})<S(p_{j_0}^{\alpha_{j_0}})$ and $S(p_{j_0}^{\alpha_{j_0}-1})\geq S(p_i^{\alpha_i})$ for $i\neq i_0,j_0$ we also suppose that $S(p_{j_0}^{\alpha_{j_0}})=\max\left\{S(p_j^{\alpha_j}/S(p_{i_0}^{\alpha_{i_0}-1})<S(p_j^{\alpha_j})\right\}$.

Then

$$S(n)=S(p_{i_0}^{\alpha_{i_0}})(1-C_{t-1}^{1}+C_{t-1}^{2}\ldots+(-1)^{t-1}C_{t-1}^{t-1})+$$
$$+S(p_{j_0}^{\alpha_{j_0}})(-1+C_{t-2}^{1}-C_{t-2}^{2}-\ldots+(-1)^{t-1}C_{t-2}^{t-2})+$$
$$+S(p_{j_0}^{\alpha_{j_0}-1})(1-C_{t-2}^{1}+C_{t-2}^{2}-\ldots+(-1)^{t-2}C_{t-2}^{t-2})$$

so $S(n)=0$ if $t\geq 3$ or $S(p_{j_0}^{\alpha_{j_0}-1})=S(p_{j_0}^{\alpha_{j_0}})$ and $S(n)=-p_{j_0}$ otherwise.

Consequently, to obtain $S(n)$ we construct as above a maximal sequence

$i_1,i_2,\ldots,i_k$, so that $S(n)=S(p_{i_1}^{\alpha_{i_1}}), S(p_{i_1}^{\alpha_{i_1}-1})<S(p_{i_2}^{\alpha_{i_2}}),\ldots,S(p_{i_{k-1}}^{\alpha_{i_{k-1}}-1})<S(p_{i_k}^{\alpha_{i_k}})$ and it results that $S(n)=0$ if $t\geq k+1$ or $S(p_{i_k}^{\alpha_{i_k}})=\overline{S}(p_{i_k}^{\alpha_{i_k}-1})$ and $S(n)=(-1)^{k+1}$ otherwise.

Let us observe that

$$S(p^{\alpha})=S(p^{\alpha-1})\Leftrightarrow(p-1)\alpha+\sigma_{[p]}(\alpha)=(p-1)(\alpha-1)+\sigma_{[p]}(\alpha-1)\Leftrightarrow$$
$$\Leftrightarrow\sigma_{[p]}(\alpha-1)-\sigma_{[p]}(\alpha)=p-1$$

Otherwise we have $\sigma_{[p]}(\alpha-1)-\sigma_{[p]}(\alpha)=-1$. So we may write

$$S(n)=\begin{cases}0 & if\ t\geq k+1\ or\ \sigma_{[p]}(\alpha_k-1)-\sigma_{[p]}(\alpha_k)=p-1\\ (-1)^{k+1}p_k & otherwise\end{cases}$$

Application. It is said [10] that if $(V,\bigwedge,\bigvee)$ is a finit lattice, with the induced order $\leq$ and for the function $f:V\rightarrow N$ we consider the generating function F defined as in (15) then if $g_{ij}=F(x_i\bigwedge x_j)$ it results det $g_{ij}=f(x_1).f(x_2)..f(x_n)$. In [10] it is showen also that this assertion may be generalized for partial ordered set by defining

$$g_{ij}=\sum_{\substack{x\leq x_i\\ x\leq x_j}}f(x)$$

Using these results if we denote by (i,j) the greatest common divisor of i and j , end $\Delta(r)=\det(S((i,j)))$ for $i,j=\overline{1,r}$ then $\Delta(r)=s(1).s(2)....s(r)$. That is for a sufficient large we have $\Delta(r)=0$ (in fact for $r\geq 8$). Moreover, for every n there exists a sufficient large r so that $\Delta(n,r)=\det(S(n+i,n+j)))=0$, for $i,j=\overline{1,r}$ because

$$\Delta(n,r)=\prod_{i=1}^{n}S(n+1)\ .$$

4. The extention of S to the rational numbers

To obtain this extention we shall define first a dual function of the Smarandache function.

In [4] and [6] a dulity principale is used to obtain, starting from a given lattice on the unit interval, other lattices on the some set. The results are used to propose a definition of bitopological spaces and to introduce a new point of view for studying the fuzzy sets. In [5] the method to obtain news lattices on the unit, interval is generalised for an arbitrary lattice.

In the following we adopt a method from [5] to construct all the functions tied in a certain sense by duality to the Smarandache function.

Let us observe that if we note

$$\mathfrak{R}_d(n)=\{m/n\leq_d m!\}\ ,\ \mathfrak{L}_d(n)=\{m/m!\leq_d n\}\ ,\ \mathfrak{R}(n)=\{m/n\leq m!\}\ ,\ \mathfrak{L}(n)=\{m/m!\leq n\}$$

then we may say that the function S is defined by the triplet $(\bigwedge,\in,\mathfrak{R}_d)$, because $S(n)=\bigwedge\{m/m\in\mathfrak{R}_d(n)\}$ .Now we may investigate all the functions defined by means of a triplet (a,b,c), where a is one of the symbols $\bigvee,\bigwedge,\overset{d}{\bigvee},\underset{d}{\bigwedge}$, b is one of the symbols $\in,\notin$ and c is one of the sets $\mathfrak{R}_d(n),\mathfrak{L}_d(n),\mathfrak{R}(n),\mathfrak{L}(n)$ defined above.

Not all of these function are non-trivial. As we have already seen the triplet $(\bigwedge,\in,\mathfrak{R}_d)$ defines the function $S_1(n)=S(n)$, but the triplet $(\bigwedge,\in,\mathfrak{L}_d)$ defines the functions $S_2(n)=\bigwedge\{m/m!\leq_d n\}$, which is identically one.

Many of the functions obtained by this method are step functions. For instance let S_3 be the function defined by $(\bigwedge,\in,\mathfrak{R})$. We have $S_3(n)=\bigwedge\{m/n\leq m!\}$ so $S_3(n)=m$ if and only if $n\in[(m-1)!+1,m!]$. Let us focus the attention on the function defined by $(\bigvee,\in,\mathfrak{L}_d)$

$$S_4(n)=\bigvee\{m/m!\leq_d n\} \quad (21)$$

where thereis in a certain sense the dual of Smarandache function.

Proposition 4.1. The function S_4 satisfies

$$S_4(n_1 \underset{d}{\wedge} n_2)=S_4(n_1)\wedge S_4(n_2) \quad (22)$$

so is a morphism from $(N^*, \underset{d}{\wedge})$ to $(N^*, \wedge)$.

Proof. Let us denote by $p_1, p_2, \ldots, p_i, \ldots$ the sequence of the prime numbers and let $n_1=\prod p_i^{\alpha_i}, n_2=\prod p_i^{\beta_i}$.

The $n_1 \underset{d}{\wedge} n_2=\prod p_i^{\min(\alpha_i,\beta_i)}$, $S_4(n_1 \underset{d}{\vee} n_2)=m$, $S_4(n_i)=m_i$, for i=1,2 and we supose $m_1\leq m_2$ then the right hand in (22) is $m_1\wedge m_2=m$.

By the definition S_4 we have $E_{p_i}(m)\leq \min(\alpha_i,\beta_i)$ for $i\geq 1$ and there exists j so that $E_{p_j}(m+1)>\min(\alpha_j,\beta_j)$. Then $\alpha_i \geq E_{p_i}(m)$ and $\beta_i\geq E_{p_i}(m)$ for al $i\geq 1$. We also have $E_{p_i}(m_r)\leq \alpha_i$ for r=1,2. In addition there exists h and k so that $E_{p_h}(m_1+1)>\alpha_h$, $E_{p_j}(m_2+1)>\alpha_k$.

Then $\min(\alpha_i,\beta_i)\geq \min(E_{p_i}(m_1)$, $E_{p_i}(m_2))=E_{p_i}(m_1)$, because $m_1\leq m_2$, so $m_1\leq m_2$. If we assume $m_1<m_2$ it results that $m!\leq n_1$ so it exists h so that $E_{p_h}(m)>\alpha_h$ and we have the contradiction $E_{p_h}(m)>\min\{\alpha_h,\beta_h\}$. Of course

$S_4(2n+1)=1$ and

$$S_4(n)>1 \quad \text{if and only if n is even} \qquad (23)$$

Proposition 4.2. Let $p_1, p_2, \ldots, p_i, \ldots$ be the sequence all consecutive primes and

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_k^{\alpha_k} \cdot q_1^{\beta_1} \cdot q_2^{\beta_2} \ldots q_r^{\beta_r}$$

the decomposition of $n \in N^*$ into primes such that the first part of the decomposition contains the (eventualy) consecutive primes and let

$$t_i = \begin{cases} S(p_i^{\alpha_i}) - 1 & \text{if } E_{p_i}(S(p_i^{\alpha_i})) > \alpha_i \\ S(p_i^{\alpha_i}) + p_i - 1 & \text{if } E_{p_i}(S(p_i^{\alpha_i})) = \alpha_i \end{cases} \qquad (24)$$

then $S_n(n) = \min\{t_1, t_2, \ldots, t_k, p_{k+1} - 1\}$

Prof. If $E_{p_i}(S(p_i^{\alpha_i})) > \alpha_i$ then from the definition of the function S it results that $S(p_i^{\alpha_i}) - 1$ is the greatest positive integer m such that $E_{p_i}(m) \le \alpha_i$ .Also if $E_{p_i}(S(p_i^{\alpha_i})) = \alpha_i$ then $S(p_i^{\alpha_i}) + p_i - 1$ is the greatest integer m with the property that $E_{p_i}(m) = \alpha_i$

It results that $\min\{t_1, t_2, \ldots, t_k, p_{k-1} - 1\}$ is the greatest integer m such that $E_{p_i}(m!) \le \alpha_i$ for i=1,2,...k.

Proposition 4.3. The function S_4 satisfies

$$S_4((n_1+n_2)) \wedge S_4([n_1, n_2]) = S_4(n_1) \wedge S_4(n_2)$$

for all positive integers n_1 and n_2

Proof. The equality results using (22) from the fact that $(n_1+n_2,[n_1,n_2])=(n_1,n_2)$.

We point out now some morphisme properties of the functions defined by a triplet (a,b,c) as above.

Proposition 4.4.(I) The functions $S_5:N^*\to N^*, S_5(n)=\overset{d}{\bigvee}\{m/m!\leq_d n\}$ satisfies

$$S_5(n_1\underset{d}{\bigwedge} n_2)=S_5(n_1)\underset{d}{\bigwedge} S_5(n_2)=S_5(n_1)\bigwedge S_5(n_2) \quad (25)$$

(ii) The function $S_6:N^*\to N^*, S_6(n)=\overset{d}{\bigvee}\{m/n\leq_d m!\}$ satisfies

$$S_6(n_1\overset{d}{\bigvee} n_2)=S_6(n_1)\overset{d}{\bigvee} S_6(n_2) \quad (26)$$

(iii) The function $S_7:N^*\to N^*, S_7(n)=\overset{d}{\bigvee}\{m/m!\leq n\}$ satisfies

$$\begin{aligned} S_7(n_1\bigwedge n_2)&=S_7(n_1)\bigwedge S_7(n_2)\\ S_7(n_1\bigvee N_2)&=S_7(n_1)\bigvee S_7(n_2)\end{aligned} \quad (27)$$

Proof. (I) Let $A=\{a_i/a_i!\leq_d n_1\}$, $B=\{b_j/b_j!\leq_d n_2\}$, $C=\left\{c_k/c_k!\leq_d n_1\underset{d}{\bigwedge} n_2\right\}$

Then we have $A\subset B$ or $B\subset A$. Indeed, let $A=\{a_1,a_2,...,a_h\}$, $B=\{b_1,b_2,...,b_r\}$ so that

$a_i<a_{i+1}$ and $b_j<b_{j+1}$. Then if $a_h<b_r$ it results that $a_i<b_r$ for $i=\overline{1,h}$ so

$a_i!\leq_d b_r!\leq_d n_2$. That minds $A\subset B$. Analogously, if $b_r\leq a_h$ it results $B\subset A$. Of

course we have $C=A\cap B$ so if $A\subset B$ it results

$$S_5(n_1 \underset{d}{\wedge} n_2) = \overset{d}{\vee} c_k = \overset{d}{\vee} a_i = S_5(n_1) = \min((S_5(n_1), S_5(n_2)) = S_5(n_1) \underset{d}{\wedge} S_5(n_2)$$

From (25) it results that S_5 is order preserving in $\mathcal{L}_d$ (but not in $\mathcal{L}$ because $m! < m!+1$ but $S_5(m!) = [1,2,...,m]$ and $S_5(m!+1) = 1$, because $m!+1$ is odd).

(ii) Let us observe that $S_6(n) = \overset{d}{\vee} \{m / \exists\ i \in \overline{1,t}\ \text{so that}\ E_{p_i}(m) < \alpha_i\}$.

If $a = \vee \{m / n \leq_d m!\}$ then $n \leq_d (a+1)!$ and $a+1 = \wedge \{m / n \leq_d m!\} = S(n)$,so $S_6(n) = [1,2,...,S(n)-1]$.

Thenn we have

$$S_6(n_1 \overset{d}{\vee} n_2) = [1,2,...,S(n_1 \overset{d}{\vee} n_2) - 1] = [1,2,...,S(n_1) \vee S(n_2) - 1]$$

and

$$S_6(n_1) \overset{d}{\vee} S_6(n_2) = [[1,2,...,S_6(n_1)-1],[1,2,...,S_6(n_2)-1]] = [1,2,...,S_6(n_1) \vee S_6(n_2) - 1]$$

(iii) The relations (27) results from the fact that $S_7(n) = [1,2,...,m]$ if and only if $n \in [m!, (m+1)!-1]$.

Now we may extend the Smarandache function to the rational numbers. Every positive rational number a possesses a unique prime decomposition of the form

$$a = \prod_p p^{\alpha_p} \quad (28)$$

with integral exponents α_p of which only finitely many are nonzero. Multiplication of rational numbers is reduced to addition of their integral exponent systems. As a consequence of this reduction questions concerning divisibility of rational numbers are reduced to questions concerning ordering of the corresponding exponent systems. That is if $b=\prod_p p^{\beta_r}$ then b divides a if and only if $\beta_p \le \alpha_p$ for all p. The greatest common divisor d and the least common multiple e are given by

$$\begin{aligned} d&=(a,b,\dots)=\prod_p p^{\min(\alpha_p,\beta_p,\dots)} \\ e&=[a,b,\dots]=\prod_p p^{\max(\alpha_p,\beta_p,\dots)} \end{aligned} \quad (29)$$

Furthermore, the least common multiple of nonzero numbers (multiplicatively bounded above) is reduced by the rule

$$[a,b,\dots]=\frac{1}{(\frac{1}{a},\frac{1}{b},\dots)} \quad (30)$$

to the greatest common divisor of their reciprocals (multiplicatively bounded below).

Of course we may write every positive rational a under the form $a=n/n_1$ with n and n_1 positive integers.

Definition 4.5. The extention $S:Q_+^* \to Q_+^*$ of the Smarandache function is defined by

$$S(\frac{n}{n_1})=\frac{S_1(n)}{S_4(n_1)} \quad (31)$$

A consequence of this definition is that if n_1 and n_2 are positive integers then

$$S(\frac{1}{n_1} \overset{d}{\vee} \frac{1}{n_2})=S(\frac{1}{n_1}) \vee S(\frac{1}{n_2}) \quad (32)$$

Indeed

$$S(\frac{1}{n_1}\overset{d}{\bigvee}\frac{1}{n_2})=S(\frac{1}{n_1\underset{d}{\bigwedge}n_2})=\frac{1}{S_4(n_1\underset{d}{\bigwedge}n_2)}=\frac{1}{S_4(n_1)\bigwedge S_4(n_2)}=$$

$$=\frac{1}{S_4}(n_1)\bigvee\frac{1}{S_4}(n_2)=S(\frac{1}{n_1})\bigvee S(\frac{1}{n_2})$$

and we can imediatly deduce that

$$S(\frac{n}{n_1}\overset{d}{\bigvee}\frac{m}{m_1})=(S(n)\bigvee S(m)).(S(\frac{1}{n_1})\bigvee(\frac{1}{m_1}))\quad(33)$$

It results that the function $\tilde{S}$ defined by $\tilde{S}(a)=\frac{1}{S(\frac{1}{a})}$ satisfies

$$\tilde{S}(n_1\underset{d}{\bigwedge}n_2)=\tilde{S}(n_1)\bigwedge\tilde{S}(n_2)$$

and

$$\tilde{S}(\frac{1}{n_1}\underset{d}{\bigwedge}\frac{1}{n_2})=\tilde{S}(\frac{1}{n_1})\bigwedge\tilde{S}(\frac{1}{n_2})\quad(34)$$

for every positive integers n_1 and n_2. Moreover, it results that

$$\tilde{S}(\frac{n_1}{m_1}\underset{d}{\bigwedge}\frac{n_2}{m_2})=(\tilde{S}(n_1)\bigwedge\tilde{S}(n_2)).(\tilde{S}(\frac{1}{m_1})\bigwedge\tilde{S}(\frac{1}{m_2}))$$

and of course the restriction of $\tilde{S}$ to the positive integers is S_4. The extention of S to all the rationals is given by $S(-a)=S(a)$.

5. Numerical function inspired from the definition of the Smarandache function

We shall use now the equality (21) and the relation (18) to consider numerical functions as the Smarandache function.

We may say that m! Is the product of all positive "smaller" than m in the lattice $\mathcal{L}$. Analogously the product ρ_m of all the divizors of m is the product of all the elements "smaller" than m in the lattice $\mathcal{L}$. So we may consider functions of the form

$$\theta(n)=\bigwedge\{m/n\leq_d\rho(m)\} \qquad (35)$$

It is said that if $m=p_1^{x_1}\cdot p_2^{x_2}\cdot\ldots\cdot p_t^{x_t}$ then the product of all the divisors of m is $\rho(m)=\sqrt{m^{\tau(m)}}$ were $\tau(m)=(x_1+1)(x_2+1)\ldots(x_t+1)$ is the number of all the divisors of m.

If n is given as in (1) then $n\leq_d\rho(m)$ if and only if:

$$g_1\equiv x_1(x_1+1)(x_2+1)\ldots(x_t+1)-2\alpha_1\geq 0$$

$$g_2\equiv x_2(x_1+1)(x_2+1)\ldots(x_t+1)-2\alpha_2\geq 0$$

....

$$g_t\equiv x_t(x_1+1)(x_2+1)\ldots(x_t+1)-2\alpha_t\geq 0 \qquad (37)$$

so $\theta(n)$ may be obtaine solving the problem of non linear programing

$$(\min)f=p_1^{x_1}p_2^{x_2}\ldots p_t^{x_t} \qquad (38)$$

under the restrictions (37).

The solutions of this problem may be obtaine applying the algorithm SUMT (Sequential Unconstrained Minimization Techniques) does to Fiacco and Mc Cormick

[7].

Examples

1^0 For $n=3^4\cdot5^{12}$, (37) and (38) become (min) $f(x)=3^x15^{x_2}$ with $x_1(x_1+1)(x_2+1)\geq 8$, $x_2(x_1+1)(x_2+1)\geq 24$. Considering the function

$U(x,n)=f(x)-r\sum_{i=1}^{n}\ln g_i(x)$, and the system

$$\delta U/\delta x_1=0\ ,\quad \delta U/\delta x_2=0 \qquad (39)$$

in [7] it is showed that if the solution $x_1(r),x_2(r)$ can't be explained from the system we can make $r\to0$. Then the system becomes $x_1(x_1+1)(x_2+1)=8,x_2(x_1+1)(x_2+1)=24$ with the (real) solution $x_1=1,x_2=3$.

So, we have $\min\{m/3^4\cdot5^{12}\leq\rho(m)\}=m_0=3\cdot5^3$

Indeed $\rho(m_0)=m_0^{\tau(m_0)/2}=m_0^1=3^4\cdot5^{12}=n$

2^0 For $n=3^2\cdot5^7$, from the system (39) it results for x_2 the equation $2x_2^3+9x_2^2+7x_2+98=0$, with the real solution $x_2\in(2,3)$. It results $x_1\in(4/7,5/7)$. Considering $x_1=1$, we observe that for $x_2=2$ the problem, but $x_2=3$ give $\theta(3^2\cdot5^7)=3^4\cdot5^{12}$.

3^0 Generaly for $n=p_1^{\alpha_1}\cdot p_2^{\alpha_2}$, from the system (39) it results the equation

$$\alpha_1x_2^3+(\alpha_1+\alpha_2)\cdot x_2^2+\alpha_2x_2-2\alpha_2^2=0$$

with solutions given by Cartan's formula.

Of course, using "the method of the triplets", as for the Smarandache function, many other functions may be associated to θ .

For the function ν given by (18) it is also possibile to generate a class of function by means of such triplets.

In the sequel we'll focus the attention on the analogous of the Smarandache function and on his dual in this case.

Proposition 5.1. If in has the decomposition into primes given by (1) then

(i) $\nu(n)=\max\limits_{i=\overline{1,t}} p_i^{\alpha_i}$

(ii) $\nu(n_1 \overset{d}{V} n_2)=\nu(n_1) V \nu(n_2)$

Proof.

(i) Let be $\max p_i^{\alpha_1}=p_u^{\alpha_u}$. Then $p_i^{\alpha_i}\leq p_u^{\alpha_u}$ for all $i=\overline{1,t}$, so $p_i^{\alpha_i}\leq_d [1,2,\ldots,p_u^{\alpha_u}]$.

But $(p_i^{\alpha_i},p_j^{\alpha_j})$ for $i\neq j$ and then $n\leq_d [1,2,\ldots,p_u^{\alpha_u}]$

Now if for some $m<p_u^{\alpha_u}$ we have $n\leq_d [1,2,\ldots,m]$, it results the contradiction

$p_u^{\alpha_u}\leq_d [1,2,\ldots,m]$.

(ii) If $n_1=\Pi p^{\alpha_p}$, $n_2=\Pi p^{\beta_p}$ then $n_1 \overset{d}{V} n_2=\Pi p^{\max(\alpha_p,\beta_p)}$ so

$$\nu(n_1 \overset{d}{\bigvee} n_2)=\max p^{\max(\alpha_p,\beta_p)}=\max(\max p^{\alpha_p},\max p^{\beta_p}) .$$

The function $\nu_1=\nu$ is defined by means of the triplet $(\bigwedge,\in\Re_{[d]})$ where $\Re_{[d]}=\{m/n\leq_d [1,2,\ldots,m]\}$. His dul, in the sense of the above section is the function defined by the triplet $(\bigvee,\in,\mathfrak{L}_{[d]})$. Let us note this function

$$v_4(n)=\bigvee\{m/[1,2,...,m]\leq_d n\}$$

That is $v_4(n)$ is the greatest natural number with the property that all $m\leq v_4(n)$ divide n.

Let us observe that a necessary and sufficient condition to have $v_4(n)>1$ is to exists $m>1$ so that every prime $p\leq m$ divide n. From the definition of v_4 it is also results that $v_4(n)=m$ if and only if n is divisible by every $i\leq n$ and note by $m+1$.

Proposition 5.2. The function v_4 satisfies

$$v_4(n_1 \underset{d}{\bigwedge} n_2)=v_4(n_1)\bigwedge v_4(n_2)$$

Proof. Let us note $n=n_1 \underset{d}{\bigwedge} n_2$, $v_4(n)=m$, $v_4(n_i)=m_i$ for i=1,2. If $m_1=m_1\bigwedge m_2$ then we prove that $m=m_1$. From the definition of v_4 it results

$$v_4(n_i)=m_i \Leftrightarrow [\forall i\leq m_i \rightarrow n \text{ is divisible by i but not by m+1}]$$

If $m<m_1$ then $m+1\leq m_1\leq m$ so $m+1$ divides n_1 and n_2 . That is m+1 divides n.

If $m>m_1$ then $m_1+1\leq n$ so m_1+1 divides n. But n divides n_1 , so m_1+1 divides n_1 .

If $t_0=\max\{i/j\leq i\Rightarrow n \textit{ is divide } n\}$ then $v_4(n)$ may be obtained solving the integer linear programming problem

$$\begin{aligned}&(\max) f=\sum_{i=1}^{t_0} x_i \ln p\\&x_i \leq \alpha_i \text{ for } i=\overline{1,t_0} \quad (41)\\&\sum_{x=1}^{t_0} x_i \ln p_i \leq \ln p_{t_0+1}\end{aligned}$$

If f_0 is the maximal value of f for above problem, then $v_4(n)=e^{f_0}$

For instance $v_4(2^3.3^2.5.11)=6$

Of course, the function v may be extinded to the rational numbers in the same way as the Smarandache function.

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