GENERALIZATION OF THE THEOREM OF MENELAUS USING A SELF-RECURRENT METHOD

Florentin Smarandache
University of New Mexico, Gallup Campus, USA

Abstract.
This generalization of the Theorem of Menelaus from a triangle to a polygon with \( n \) sides is proven by a self-recurrent method which uses the induction procedure and the Theorem of Menelaus itself.

The Theorem of Menelaus for a Triangle is the following:

If a line \((d)\) intersects the triangle \( \Delta A_1A_2A_3 \) sides \( A_1A_2, A_2A_3, \) and \( A_3A_1 \) respectively in the points \( M_1, M_2, M_3, \) then we have the following equality:

\[
\frac{M_1A_1}{M_1A_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot \frac{M_3A_3}{M_3A_1} = 1
\]

where by \( M_iA_i \) we understand the (positive) length of the segment of line or the distance between \( M_i \) and \( A_i \); similarly for all other segments of lines.

Let’s generalize the Theorem of Menelaus for any \( n \)-gon (a polygon with \( n \) sides), where \( n \geq 3 \), using our Recurrence Method for Generalizations, which consists in doing an induction and in using the Theorem of Menelaus itself.

For \( n = 3 \) the theorem is true, already proven by Menelaus.

The Theorem of Menelaus for a Quadrilateral.

Let’s prove it for \( n = 4 \), which will inspire us to do the proof for any \( n \).

Suppose a line \((d)\) intersects the quadrilateral \( A_1A_2A_3A_4 \) sides \( A_1A_2, A_2A_3, A_3A_4, \) and \( A_4A_1 \) respectively in the points \( M_1, M_2, M_3, \) and \( M_4 \), while its diagonal \( A_2A_4 \) into the point \( M \) [see Fig. 1 below].

We split the quadrilateral \( A_1A_2A_3A_4 \) into two disjoint triangles \( (3\text{-gons}) \) \( \Delta A_1A_2A_3 \) and \( \Delta A_4A_2A_3 \), and we apply the Theorem of Menelaus in each of them, respectively getting the following two equalities:
and

\[ \frac{M_1A_1}{M_1A_2} \cdot \frac{MA_2}{MA_4} \cdot \frac{M_4A_4}{M_4A_1} = 1 \]

Now, we multiply these last two relationships and we obtain the Theorem of Menelaus for \( n = 4 \) (a quadrilateral):

\[ \frac{M_1A_1}{M_1A_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot \frac{M_3A_3}{M_3A_4} \cdot \frac{M_4A_4}{M_4A_1} = 1. \]

Let’s suppose by induction upon \( k \geq 3 \) that the Theorem of Menelaus is true for any \( k \)-gon with \( 3 \leq k \leq n - 1 \), and we need to prove it is also true for \( k = n \).

Suppose a line \((d)\) intersects the \( n \)-gon \( A_1A_2\ldots A_n \) sides \( A_iA_{i+1} \) in the points \( M_i \), while its diagonal \( A_2A_n \) into the point \( M \) (of course by \( A_nA_{n+1} \) one understands \( A_nA_1 \)) – see Fig. 2.

We consider the \( n \)-gon \( A_1A_2\ldots A_{n-1}A_n \) and we split it similarly as in the case of quadrilaterals in a \( 3 \)-gon \( \Delta A_1A_2A_n \) and an \((n-1)\)-gon \( A_nA_2A_3\ldots A_{n-1} \) and we can respectively apply the Theorem of Menelaus according to our previously hypothesis of induction in each of them, and we respectively get:

\[ \frac{M_1A_1}{M_1A_2} \cdot \frac{MA_2}{MA_n} \cdot \frac{M_nA_n}{M_nA_1} = 1 \]

and
whence, by multiplying the last two equalities, we get

the **Theorem of Menelaus for any n-gon:**

\[
\frac{MAn}{MA2} \cdot \frac{M2A2}{M2A3} \cdots \frac{Mn-2An-2}{Mn-2An-1} \cdot \frac{Mn-1An-1}{Mn-1An} = 1
\]

\[
\prod_{i=1}^{n} \frac{M: Ai}{M: Ai+1} = 1.
\]

**Fig. 2**

**Conclusion.**

We hope the reader will find useful this self-recurrence method in order to generalize known scientific results by means of themselves!

*{Translated from French by the Author.}*

**References:**


