A GENERALIZATION OF A THEOREM OF CARNOT

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Theorem of Carnot: Let \( M \) be a point on the diagonal \( AC \) of an arbitrary quadrilateral \( ABCD \). Through \( M \) one draws a line which intersects \( AB \) in \( \alpha \) and \( BC \) in \( \beta \). Let us draw another line, which intersects \( CD \) in \( \gamma \) and \( AD \) in \( \delta \). Then one has:

\[
\frac{A\alpha}{B\beta} \cdot \frac{B\beta}{C\gamma} \cdot \frac{C\gamma}{D\gamma} \cdot \frac{D\delta}{A\delta} = 1.
\]

Generalization: Let \( A_1...A_n \) be a polygon. On a diagonal \( A_iA_k \) of this polygon one takes a point \( M \) through which one draws a line \( d_1 \) which intersects the lines \( A_1A_2, A_2A_3, ..., A_{k-1}A_k \) respectively in the points \( P_1, P_2, ..., P_{k-1} \) and another line \( d_2 \) intersects the other lines \( A_kA_{k+1}, ..., A_nA_1 \) respectively in the points \( P_k, ..., P_{n-1}, P_n \). Then one has:

\[
\prod_{i=1}^{n} \frac{A_iP_i}{A_{\varphi(i)}P_i} = 1,
\]

where \( \varphi \) is the circular permutation

\[
\begin{pmatrix}
1 & 2 & \ldots & n-1 & n \\
2 & 3 & \ldots & n & 1
\end{pmatrix}.
\]

Proof:
Let us have \( 1 \leq j \leq k - 1 \). One easily shows that:

\[
\frac{A_jP_j}{A_{j+1}P_j} = \frac{D(A_j, d_1)}{D(A_{j+1}, d_1)}
\]

where \( D(A, d) \) represents the distance from the point \( A \) to the line \( d \), since the triangles \( P_jA_jA_j' \) and \( P_jA_{j+1}A_{j+1}' \) are similar. (One notes with \( A_j' \) and \( A_{j+1}' \) the projections of the points \( A_j \) and \( A_{j+1} \) on the line \( d_1 \)).

It results from it that:

\[
\frac{A_1P_1}{A_2P_1} \cdot \frac{A_2P_2}{A_3P_2} \cdot \ldots \cdot \frac{A_{k-1}P_{k-1}}{A_kP_{k-1}} = \frac{D(A_1, d_1)}{D(A_2, d_1)} \cdot \frac{D(A_2, d_1)}{D(A_3, d_1)} \cdot \ldots \cdot \frac{D(A_{k-1}, d_1)}{D(A_k, d_1)} = \frac{D(A_1, d_1)}{D(A_k, d_1)}
\]

In a similar way, for \( k \leq h \leq n \) one has:
\[
\frac{A_h P_h}{A_{\phi(h)} P_h} = \frac{D(A_h, d_2)}{D(A_{\phi(h)}, d_2)}
\]

and

\[
\prod_{h=1}^{n} \frac{A_h P_h}{A_{\phi(h)} P_h} = \frac{D(A_1, d_2)}{D(A_1, d_2)}
\]

The product of the theorem is equal to:

\[
\frac{D(A_1, d_1)}{D(A_k, d_1)} \cdot \frac{D(A_k, d_2)}{D(A_1, d_2)}
\]

but

\[
\frac{D(A_1, d_1)}{D(A_k, d_1)} = \frac{A_1 M}{A_k M}
\]

since the triangles \( MA_1 A'_1 \) and \( MA_k A'_k \) are similar. In the same way, because the triangles \( MA_i A'_i \) and \( MA_k A'_k \) are similar (one notes with \( A'_i \) and \( A'_k \) the respective projections of \( A_i \) and \( A_k \) on the line \( d_2 \)), one has:

\[
\frac{D(A_k, d_2)}{D(A_1, d_2)} = \frac{A_k M}{A_1 M}
\]

The product from the statement is therefore equal to 1.

Remark: If one replaces \( n \) by 4 in this theorem, one finds the theorem of Carnot.