GENERALIZATIONS OF THE THEOREM OF CEVA

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In these paragraphs one presents three generalizations of the famous theorem of Ceva, which states:

"If in a triangle ABC one plots the convergent straight lines

AA_1, BB_1, CC_1 then $\frac{\overline{A_1B}}{\overline{A_1C}} \cdot \frac{\overline{B_1C}}{\overline{B_1A}} \cdot \frac{\overline{C_1A}}{\overline{C_1B}} = -1$".

Theorem: Let us have the polygon $A_1A_2...A_n$, a point M in its plane, and a circular permutation

$p = \begin{pmatrix} 1 & 2 & ... & n-1 & n \\ 2 & 3 & ... & n & 1 \end{pmatrix}$. One notes M_{ij} the intersections of the line A_iM with the lines $A_{i+s}A_{i+s+1}, ..., A_{i+s+t-1}A_{i+s+t}$ (for all i and j, $j \in \{i+s, ..., i+s+t-1\}$).

If $M_{ij} \neq A_n$ for all the respective indices, and if $2s + t = n$, one has:

$\prod_{i,j=1,i+s}^{n,i+s+t-1} \frac{\overline{M_{ij}A_j}}{\overline{M_{ij}A_{p(j)}}} = (-1)^n$ (s and t are natural non zero numbers).

Analytical demonstration: Let M be a point in the plain of the triangle ABC, such that it satisfies the conditions of the theorem. One chooses a Cartesian system of axes, such that the two parallels with the axes which pass through M do not pass by any point A_i (this is possible).

One considers $M(a,b)$, where a and b are real variables, and $A_i(X_i, Y_i)$ where X_i and Y_i are known, $i \in \{1, 2, ..., n\}$.

The former choices ensure us the following relations:

$X_i - a \neq 0$ and $Y_i - b \neq 0$ for all $i \in \{1, 2, ..., n\}$.

The equation of the line A_iM $(1 \leq i \leq n)$ is:

$\frac{x-a}{X_i - a} - \frac{y-b}{Y_i - b}$. One notes that $d(x, y; X_i, Y_i) = 0$.

One has

$$\frac{\overline{M_{ij}A_j}}{\overline{M_{ij}A_{p(j)}}} = \frac{\delta(A_j, A_iM)}{\delta(A_{p(j)}, A_iM)} = \frac{d(X_j, Y_j; X_i, Y_i)}{d(X_{p(j)}, Y_{p(j)}; X_i, Y_i)} = \frac{D(j,i)}{D(p(j),i)}$$

where $\delta(A, ST)$ is the distance from A to the line ST, and where one notes with $D(a,b)$ for $d(X_a, Y_a; X_b, Y_b)$.

Let us calculate the product, where we will use the following convention: $a+b$ will mean $\underbrace{p(p(...p}_{\text{b times}}(a)...))$, and $a-b$ will mean $\underbrace{p^{-1}(p^{-1}(...p^{-1}}_{\text{b times}}(a)...))$

$$\prod_{j=i+s}^{i+s+t-1} \frac{\overline{M_{ij}A_j}}{\overline{M_{ij}A_{j+1}}} = \prod_{j=i+s}^{i+s+t-1} \frac{D(j,i)}{D(j+1,i)} =$$

$$= \frac{D(i+s,i)}{D(i+s+1,i)} \cdot \frac{D(i+s+1,i)}{D(i+s+2,i)} \cdots \frac{D(i+s+t-1,i)}{D(i+s+t,i)} =$$

$$= \frac{D(i+s,i)}{D(i+s+t,i)} = \frac{D(i+s,i)}{D(i-s,i)}$$

The initial product is equal to:

$$\prod_{i=1}^{n} \frac{D(i+s,i)}{D(i-s,i)} = \frac{D(1+s,1)}{D(1-s,1)} \cdot \frac{D(2+s,2)}{D(2-s,2)} \cdots \frac{D(2s,s)}{D(n,s)} \cdot$$

$$\cdot \frac{D(2s+2,s+2)}{D(2,s+2)} \cdots \frac{D(2s+t,s+t)}{D(t,s+t)} \cdot \frac{D(2s+t+1,s+t+1)}{D(t+1,s+t+1)} \cdot$$

$$\cdot \frac{D(2s+t+2,s+t+2)}{D(t+2,s+t+2)} \cdots \frac{D(2s+t+s,s+t+s)}{D(t+s,s+t+s)} =$$

$$= \frac{D(1+s,1)}{D(1,1+s)} \cdot \frac{D(2+s,2)}{D(2,2+s)} \cdots \frac{D(2s+t,s+t)}{D(s+t,2s+t)} \cdots \frac{D(s,n)}{D(n,s)} =$$

$$= \prod_{i=1}^{n} \frac{D(i+s,i)}{D(i,i+s)} = \prod_{i=1}^{n} \left(-\frac{P(i+s)}{P(i)} \right) = (-1)^n$$

because:

$$\frac{D(r,p)}{D(p,r)} = \frac{\dfrac{X_r-a}{X_p-a} - \dfrac{Y_r-b}{Y_p-b}}{\dfrac{X_p-a}{X_r-a} - \dfrac{Y_p-b}{Y_r-b}} = -\frac{(X_r-a)(Y_r-b)}{(X_p-a)(Y_p-b)} = -\frac{P(r)}{P(p)},$$

The last equality resulting from what one notes: $(X_t-a)(Y_t-b)=P(t)$. From (1) it results that $P(t)\neq 0$ for all t from $\{1,2,...,n\}$. The proof is completed.

Comments regarding the theorem:

t represents the number of lines of a polygon which are intersected by a line $A_{i_0}M$; if one notes the sides A_iA_{i+1} of the polygon, by a_i, then $s+1$ represents the order of the first line intersected by the line A_1M (that is a_{s+1} the first line intersected by A_1M).

Example: If $s = 5$ and $t = 3$, the theorem says that :

- the line A_1M intersects the sides A_6A_7, A_7A_8, A_8A_9 .
- the line A_2M intersects the sides $A_7A_8, A_8A_9, A_9A_{10}$.
- the line A_3M intersects the sides $A_8A_9, A_9A_{10}, A_{10}A_{11}$, etc.

Observation: The restrictive condition of the theorem is necessary for the existence of the ratios $\frac{\overline{M_{ij}A_j}}{\overline{M_{ij}A_{p(j)}}}$.

Consequence 1: Let us have a polygon $A_1A_2...A_{2k+1}$ and a point M in its plan. For all i from $\{1,2,...,2k+1\}$, one notes M_i the intersection of the line $A_iA_{p(i)}$ with the line which passes through M and by the vertex which is opposed to this line. If $M_i \notin \{A_i, A_{p(i)}\}$ then one has: $\prod_{i=1}^{n} \frac{\overline{M_iA_i}}{\overline{M_iA_{p(i)}}} = -1$.

The demonstration results immediately from the theorem, since one has $s = k$ and $t = 1$, that is $n = 2k + 1$.

The reciprocal of this consequence is not true.

From where it results immediately that the reciprocal of the theorem is not true either.

Counterexample:

Let us consider a polygon of 5 sides. One plottes the lines A_1M_3, A_2M_4 and A_3M_5 which intersect in M .

Let us have $K = \frac{\overline{M_3A_3}}{\overline{M_3A_4}} \cdot \frac{\overline{M_4A_4}}{\overline{M_4A_5}} \cdot \frac{\overline{M_5A_5}}{\overline{M_5A_1}}$

Then one plots the line A_4M_1 such that it does not pass through M and such that it forms the ratio:

(2) $\frac{\overline{M_1A_1}}{\overline{M_1A_2}} = 1/K$ or $2/K$. (One chooses one of these values, for which A_4M_1 does not pass through M).

At the end one traces A_5M_2 which forms the ratio $\frac{\overline{M_2A_2}}{\overline{M_2A_3}} = -1$ or $-\frac{1}{2}$ in function of (2). Therefore the product:

$\prod_{i=1}^{5} \frac{\overline{M_iA_i}}{\overline{M_iA_{p(i)}}}$ without which the respective lines are concurrent.

Consequence 2: Under the conditions of the theorem, if for all i and $j, j \notin \{i, p^{-1}(i)\}$, one notes $M_{ij} = A_iM \cap A_jA_{p(j)}$ and $M_{ij} \notin \{A_j, A_{p(j)}\}$ then one has:

$$\prod_{i,j=1}^{n} \frac{\overline{M_{ij}A_j}}{\overline{M_{ij}A_{p(j)}}} = (-1)^n .$$

$$j \notin \{i, p^{-1}(i)\}$$

In effect one has $s = 1$, $t = n - 2$, and therefore $2s + t = n$.

Consequence 3: For $n = 3$, it comes $s = 1$ and $t = 1$, therefore one obtains (as a particular case) the theorem of Céva.