AN APPLICATION OF THE GENERALIZATION OF CEVA’S THEOREM

Florentin Smarandache
University of New Mexico
200 College Road
Gallup, NM 87301, USA
E-mail: smarand@unm.edu

Theorem: Let us consider a polygon \( A_1 A_2 \ldots A_n \) inserted in a circle. Let \( s \) and \( t \) be two non zero natural numbers such that \( 2s + t = n \). By each vertex \( A_i \) passes a line \( d_i \) which intersects the lines \( A_{i+s} \ldots, A_{i+s+1} \ldots, A_{i+s+t-1} \ldots, A_{i+s+t} \) at the points \( M_{i,s}, \ldots, M_{i+s+t-1} \) respectively and the circle at the point \( M'_i \). Then one has:

\[
\prod_{i=1}^{n} \prod_{j=i+s}^{i+s+t-1} \frac{M'_i A_j}{M'_i A_{j+1}} = \prod_{i=1}^{n} \frac{M'_i A_{i+s}}{M'_i A_{i+s+t}}.
\]

Proof:
Let \( i \) be fixed.
1) The case where the point \( M_{i,i+s} \) is inside the circle.

There are the triangles \( A_i M_{i,i+s} A_{i+s} \) and \( M'_i M_{i,i+s} A_{i+s} \) similar, since the angles \( M_{i,i+s} A_i A_{i+s} \) and \( M'_i M_{i,i+s} A_{i+s} \) are equal. It results from it that:

\[
\frac{M_{i,i+s} A_i}{M_{i,i+s} A_{i+s+1}} = \frac{A_i A_{i+s}}{M'_i A_{i+s+1}}
\]
In a similar manner, one shows that the triangles $M_{i,i+s}A_{i+s+1}$ and $M_{i,i+s}^iA_i^i$ are similar, from which:

$$\frac{M_{i,i+s}A_i}{M_{i,i+s}^iA_i^i} = \frac{A_iA_{i+s+1}}{A_i^iA_{i+s+1}}.$$ Dividing (1) by (2) we obtain:

$$\frac{M_{i,i+s}A_{i+s}}{M_{i,i+s}^iA_{i+s+1}} = \frac{M_{i,i+s}^iA_{i+s+1}}{M_{i,i+s}A_{i+s+1}} \cdot \frac{A_iA_{i+s}}{A_i^iA_{i+s+1}}.$$

2) The case where $M_{i,i+s}$ is exterior to the circle is similar to the first, because the triangles (notations as in 1) are similar also in this new case. There are the same interpretations and the same ratios; therefore one has also the relation (3).

Let us calculate the product:

$$\prod_{j=i+s}^{i+s+t-1} \frac{M_jA_j}{M_jA_{j+1}} = \prod_{j=i+s}^{i+s+t-1} \left( \frac{M_jA_j}{M_jA_{j+1}} \cdot \frac{A_jA_{j+1}}{A_jA_{j+1}} \right) =$$

$$= \frac{M_{i,s}A_{i+s}}{M_{i,i+s+1}} \cdot \frac{M_{i,s+1}A_{i+s+1}}{M_{i,i+s+2}} \cdot \frac{M_{i,s+2}A_{i+s+2}}{M_{i,i+s+3}} \cdot \frac{M_{i,s+3}A_{i+s+3}}{M_{i,i+s+4}} \cdots \frac{M_{i,s+t-1}A_{i+s+t-1}}{M_{i,i+s+t}} \cdot \frac{A_{i+s}A_{i+s+1}}{A_{i+s}A_{i+s+1}} \cdot \frac{A_{i+s+1}A_{i+s+2}}{A_{i+s+1}A_{i+s+2}} \cdots \frac{A_{i+s+t-1}A_{i+s+t-1}}{A_{i+s+t-1}A_{i+s+t}} = \frac{M_{i,s}A_{i+s}}{M_{i,i+s+1}} \frac{A_{i+s}A_{i+s+1}}{A_{i+s}A_{i+s+1}} \cdots \frac{A_{i+s+t-1}A_{i+s+t-1}}{A_{i+s+t-1}A_{i+s+t}}.$$

Therefore the initial product is equal to:
\[
\prod_{i=1}^{n} \left( \frac{M_iA_{i+s}}{M_iA_{i+s+t}} \cdot \frac{A_iA_{i+s}}{A_iA_{i+s+t}} \right) = \prod_{i=1}^{n} \frac{M_iA_{i+s}}{M_iA_{i+s+t}}
\]

since:

\[
\prod_{i=1}^{n} \frac{A_iA_{i+s}}{A_iA_{i+s+t}} = \frac{A_1A_{1+s}}{A_1A_{1+s+t}} \cdot \frac{A_2A_{2+s}}{A_2A_{2+s+t}} \cdots \frac{A_nA_{n+s}}{A_nA_{n+s+t}}.
\]

\[
\frac{A_{s+2}A_{s+2+t}}{A_{s+2}A_{s+2+t}} \cdot \frac{A_{s+1}A_{s}}{A_{s+1}A_{s+t}} \cdot \frac{A_{s+1}A_{s}}{A_{s+1}A_{s+t}} \cdots \frac{A_nA_{s}}{A_nA_{s+t}} = 1
\]

(by taking into account the fact that \(2s + t = n\)).

**Consequence 1:** If there is a polygon \(A_1A_2\ldots A_{2s-1}\) inscribed in a circle, and from each vertex \(A_i\) one traces a line \(d_i\) which intersects the opposite side \(A_{i+s-1}A_{i+s}\) in \(M_i\) and the circle in \(M'_i\) then:

\[
\prod_{i=1}^{n} \frac{M_iA_{i+s-1}}{M_iA_{i+s}} = \prod_{i=1}^{n} \frac{M'_iA_{i+s-1}}{M'_iA_{i+s}}
\]

In fact for \(t = 1\), one has \(n\) odd and \(s = \frac{n + 1}{2}\).

If one makes \(s = 1\) in this consequence, one finds the mathematical note from [1], pages 35-37.

**Application:** If in the theorem, the lines \(d_i\) are concurrent, one obtains:

\[
\prod_{i=1}^{n} \frac{M_iA_{i+s}}{M_iA_{i+s+t}} = (-1)^n
\]

(For this, see [2]).

**Bibliography:**