Generating Lemoine Circles

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In this paper, we generalize the theorem relative to the first circle of Lemoine and thereby highlight a method to build Lemoine circles. Firstly, we review some notions and results.

**Definition 1.** It is called a simedian of a triangle the symmetric of a median of the triangle with respect to the internal bisector of the triangle that has in common with the median the peak of the triangle.

**Proposition 1.** In the triangle $\triangle ABC$, the cevian $AS$, $S \in (BC)$, is a simedian if and only if $\frac{SB}{SC} = \left(\frac{AB}{AC}\right)^2$.

For Proof, see [2].

**Definition 2.** It is called a simedian center of a triangle (or Lemoine point) the intersection of triangle’s simedians.

**Theorem 1.** The parallels to the sides of a triangle taken through the simedian center intersect the triangle’s sides in six concyclic points (the first Lemoine circle - 1873).

A Proof of this theorem can be found in [2].

**Definition 3.** We assert that in a scalene triangle $\triangle ABC$ the line $MN$, where $M \in AB$ and $N \in AC$, is an anti-parallel to $BC$ if $\angle MNA \equiv \angle ABC$. 
Lemma 1. In the triangle $ABC$, let $AS$ be a simedian, $S \in (BC)$. If $P$ is the middle of the segment $(MN)$, having $M \in (AB)$ and $N \in (AC)$, belonging to the simedian $AS$, then $MN$ and $BC$ are anti-parallels.

Proof. We draw through $M$ and $N$, $MT \parallel AC$ and $NR \parallel AB$, $R,T \in (BC)$, see Figure 1. Let $\{Q\} = MT \cap NR$; since $MP = PN$ and $AMQN$ is a parallelogram, it follows that $Q \in AS$.

![Figure 1.](image-url)

Thales' Theorem provides the relations:

\[
\frac{AN}{AC} = \frac{BR}{BC} \quad (1); \quad \frac{AB}{AM} = \frac{BC}{CT} \quad (2).
\]

From (1) and (2), by multiplication, we obtain:

\[
\frac{AN}{AM} \cdot \frac{AB}{AC} = \frac{BR}{TC} \quad (3).
\]

Using again Thales' Theorem, we obtain:

\[
\frac{BR}{BS} = \frac{AQ}{AS} \quad (4), \quad \frac{TC}{SC} = \frac{AQ}{AS} \quad (5).
\]
From these relations, we get

\[
\frac{BR}{BS} = \frac{TC}{SC} \quad (6) \text{ or } \frac{BS}{SC} = \frac{BR}{TC} \quad (7).
\]

In view of Proposition 1, the relations (7) and (3) drive to \(\frac{AN}{AB} = \frac{AB}{AC}\), which shows that \(\Delta AMN \sim \Delta ACB\), so \(\propto AMN \equiv \propto ABC\), therefore \(MN\) and \(BC\) are anti-parallels in relation to \(AB\) and \(AC\).

**Remark.**

1. The reciprocal of Lemma 1 is also valid, meaning that if \(P\) is the middle of the anti-parallel \(MN\) to \(BC\), then \(P\) belongs to the simedian from \(A\).

**Theorem 2.** (Generalization of Theorem 1) Let \(ABC\) be a scalene triangle and \(K\) its simedian center. We take \(M \in AK\) and draw \(MN \parallel AB, MP \parallel AC\), where \(N \in BK, P \in CK\). Then:

i. \(NP \parallel BC\);

ii. \(MN, NP\) and \(MP\) intersect the sides of triangle \(ABC\) in six concyclic points.

**Proof.** In triangle \(ABC\), let \(AA_1, BB_1, CC_1\) the simedians concurrent in \(K\) (see Figure 2). We have from Thales' Theorem that:

\[
\frac{AM}{MK} = \frac{BN}{NK} \quad (1); \quad \frac{AM}{MK} = \frac{CP}{PK} \quad (2).
\]

From relations (1) and (2), it follows that \(\frac{BN}{NK} = \frac{CP}{PK} \quad (3)\), which shows that \(NP \parallel BC\). Let \(R, S, V, W, U, T\) be the intersection points of the parallels \(MN, MP, NP\) of the sides of the triangles to the other sides. Obviously, by construction, the quadrilaterals \(ASMW; CUPV; BRNT\) are parallelograms. The middle of the diagonal \(WS\) falls on \(AM\), so on the simedian \(AK\), and from Lemma 1 we get that \(WS\) is an anti-parallel to \(BC\). Since \(TU \parallel BC\), it follows that \(WS\) and \(TU\) are anti-parallels, therefore the points \(W, S, U, T\) are concyclic (4).
Figure 2.

Analogously, we show that the points $U, V, R, S$ are concyclic (5). From $WS$ and $BC$ anti-parallels, $UV$ and $AB$ anti-parallels, we have that $\angle WSA \equiv \angle ABC$ and $\angle VUC \equiv \angle ABC$, therefore: $\angle WSA \equiv \angle VUC$, and since $VW \parallel AC$, it follows that the trapeze $WSUV$ is isosceles, therefore the points $W, S, U, V$ are concyclic (6). The relations (4), (5), (6) drive to the concyclality of the points $R, U, V, S, W, T$, and the theorem is proved.

Remarks.

2. For any point $M$ found on the simedian $AA_1$, by performing the constructions from hypothesis, we get a circumscribed circle of the 6 points of intersection of the parallels taken to the sides of triangle.

3. The Theorem 2 generalizes the Theorem 1 because we get the second in the case the parallels are taken to the sides through the simedian center $k$. 
4. We get a circle built as in *Theorem 2* from the first Lemoine circle by homothety of pole $k$ and of ratio $\lambda \in \mathbb{R}$.

5. The centers of Lemoine circles built as above belong to the line $OK$, where $O$ is the center of the circle circumscribed to the triangle $ABC$.

**Bibliography.**
