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Integer Number Solutions of Linear Systems

INTEGER NUMBER SOLUTIONS OF LINEAR SYSTEMS

Definitions and Properties of the Integer Solution of a Linear System

Let’s consider

\[ \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = \overline{1,m} \]

a linear system with all coefficients being integer numbers (the case with rational coefficients is reduced to the same).

Definition 1. \( x_j = x_j^0, \quad j = \overline{1,n}, \) is a particular integer solution of (1) if

\[ x_j^0 \in \mathbb{Z}, \quad j = \overline{1,n} \]

and

\[ \sum_{j=1}^{n} a_{ij} x_j^0 = b_i, \quad i = \overline{1,m}. \]

Let’s consider the functions \( f_j : \mathbb{Z}^h \rightarrow \mathbb{Z}, \quad j = \overline{1,n}, \) where \( h \in \mathbb{N}^*. \)

Definition 2. \( x_j = f_j(k_1,...,k_h), \quad j = \overline{1,n}, \) is the general integer solution for (1) if:

(a) \[ \sum_{j=1}^{n} a_{ij} f_j(k_1,...,k_h) = b_i, \quad i = \overline{1,m}, \]

irrespective of \((k_1,...,k_h) \in \mathbb{Z};\)

(b) Irrespective of \( x_j = x_j^0, \quad j = \overline{1,n} \) a particular integer solution of (1) there is \((k_1^0,...,k_h^0) \in \mathbb{Z}\) such that \( f_j(k_1^0,...,k_h^0) = x_j, \quad j = \overline{1,n}. \) (In other words the general solution that comprises all the other solutions.)

Property 1.
A general solution of a linear system of \( m \) equations with \( n \) unknowns, \( r(A) = m < n, \) is undetermined \((n-m)-times.\)

Proof:
We assume by reduction ad absurdum that it is of order \( r, \) \( 1 \leq r \leq n - m \) (the case \( r = 0, \) i.e., when the solution is particular, is trivial). It follows that the general solution is of the form:

\[
\begin{align*}
    x_1 &= u_{11} p_1 + \ldots + u_{1r} p_r + v_1 \\
    \vdots \\
    x_n &= u_{n1} p_1 + \ldots + u_{nr} p_r + v_n, \quad u_{ih}, \forall i \in \mathbb{Z} \\
    p_h &= \text{parameters} \in \mathbb{Z}
\end{align*}
\]

We prove that the solution is undetermined \((n-m)-times.\)

The homogeneous linear system (1), resolved in \( r \) has the solution:
Let \( x_i = x^0_i, \ i = 1, n \), be a particular solution of the linear system (1).

Considering
\[
\begin{align*}
    x_{n+1} &= D \cdot k_{m+1} \\
    &\vdots \\
    x_n &= D \cdot k_n
\end{align*}
\]
we obtain the solution
\[
\begin{align*}
    x_1 &= D^1 \cdot k_{m+1} + \ldots + D^1 \cdot k_n + x^0_1 \\
    &\vdots \\
    x_m &= D^m \cdot k_{m+1} + \ldots + D^m \cdot k_n + x^0_m \\
    x_{m+1} &= D \cdot k_{m+1} + x^0_{m+1} \\
    &\vdots \\
    x_n &= D \cdot k_n + x^0_n, \\
    k_j &= \text{parameters } \in \mathbb{Z}
\end{align*}
\]
which depends on the \( n - m \) independent parameters, for the system (1). Let the solution be undetermined \((n - m)\)-times:
\[
\begin{align*}
    x_1 &= c_1 k_1 + \ldots + c_{n-m} k_{n-m} + d_1 \\
    &\vdots \\
    x_n &= c_n k_1 + \ldots + u_{n-m} k_{n-m} + d_n \\
    &\text{for } c_{ij}, d_i \in \mathbb{Z}, \ k_j = \text{parameters } \in \mathbb{Z}
\end{align*}
\]
(There are such solutions, we have proved it before.) Let the system be:
\[
\begin{align*}
    a_{11} x_1 + \ldots + a_{1n} x_n &= b_1 \\
    &\vdots \\
    a_{m1} x_1 + \ldots + a_{mn} x_n &= b_m
\end{align*}
\]
\( x_i = \text{unknowns } \in \mathbb{Z}, \ a_{ij}, b_j \in \mathbb{Z} \).

I. The case \( b_i = 0, \ i = 1, m \) results in a homogenous linear system:
\[
a_{11} x_1 + \ldots + a_{1n} x_n = 0; \ i = 1, m.
\]
\( (S_2) \Rightarrow a_{11} (c_1 k_1 + \ldots + c_{n-m} k_{n-m} + d_1) + \ldots + a_{mn} (c_1 k_1 + \ldots + c_{n-m} k_{n-m} + d_n) = 0 \\
0 = (a_{11} c_1 + \ldots + a_{mn} c_n) k_1 + \ldots + (a_{11} c_{1n-m} + \ldots + a_{mn} c_{n-m}) k_{n-m} + (a_{11} d_1 + \ldots + a_{mn} d_n) \\
\forall k_j \in \mathbb{Z}
\]
For \( k_1 = \ldots = k_{n-m} = 0 \Rightarrow a_{11} d_1 + \ldots + a_{mn} d_n = 0 .\)
For \( k_1 = \ldots = k_{n-m} = 0 \) and \( k_n = 1 \Rightarrow \)
\( \Rightarrow (a_i c_{i h} + \ldots + a_m c_{m h}) + (a_i d_i + \ldots + a_m d_i^{(n)}) = 0 \Rightarrow a_i c_{i h} + \ldots + a_m c_{m h} = 0, \forall \ i = 1, m, \ \forall \ h = 1, n - m. \)

The vectors
\[
V_h = \begin{pmatrix} c_{1 h} \\ \vdots \\ c_{n h} \end{pmatrix}, \quad h = 1, n - m
\]
are the particular solutions of the system.

\( V_h, \ h = 1, n - m \) also linearly independent because the solution is undetermined \( (n - m) \)-times \( \{ V_1, \ldots, V_{n-m} \} + d \) is a linear variety that includes the solutions of the system obtained from (S2).

Similarly for (S1) we deduce that
\[
U_s = \begin{pmatrix} U_{1 s} \\ \vdots \\ U_{r s} \end{pmatrix}, \quad s = 1, r
\]
are particular solutions of the given system and are linearly independent, because (S1) is undetermined \( (n - m) \)-times, and \( V = \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix} \) is a solution of the given system.

**Case (a)** \( U_1, \ldots, U_r, V = \) linearly dependent, it follows that \( \{ U_1, \ldots, U_r \} \) is a free sub-module of order \( r < n - m \) of solutions of the given system, then, it follows that there are solutions that belong to \( \{ V_1, \ldots, V_{n-m} \} + d \) and which do not belong to \( \{ U_1, \ldots, U_r \} \), a fact which contradicts the assumption that (S1) is the general solution.

**Case (b)** \( U_1, \ldots, U_r, V = \) linearly independent.
\( \{ U_1, \ldots, U_r \} + V \) is a linear variety that comprises the solutions of the given system, which were obtained from (S1). It follows that the solution belongs to \( \{ V_1, \ldots, V_{n-m} \} + d \) and does not belong to \( \{ U_1, \ldots, U_r \} + V \), a fact which is a contradiction to the assumption that (S1) is the general solution.

II. When there is an \( i \in \overline{1, m} \) with \( b_i \neq 0 \) then non-homogeneous linear system
\[
a_i x_i + \ldots + a_m x_n = b_i, \quad i = \overline{1, m}
\]
(S2) \( \Rightarrow a_i (c_{1 i} k_1 + \ldots + c_{n-m} k_{n-m} + d_i) + \ldots + a_m (c_{1 i} k_i + \ldots + c_{n-m} k_{n-m} + d_n) = b_i \)
it follows that
\[
\Rightarrow (a_i c_{1 i} + \ldots + a_m c_{m i}) k_i + \ldots + (a_i c_{1 n-m} + \ldots + a_m c_{m n-m}) k_{n-m} + (a_i d_i + \ldots + a_m d_n) = b_i
\]
For \( k_1 = \ldots = k_{n-m} = 0 \Rightarrow a_i d_i + \ldots + a_m d_n = b_i \);
For $k_1 = \ldots = k_{j-1} = k_{j+1} = \ldots = k_{n-m} = 0$ and $k_j = 1 \Rightarrow$

\[
(a_1c_{ij} + \ldots + a_nc_{nj}) + (a_1d_1 + \ldots + a_n d_n) = b_j \text{ it follows that}
\]

\[
\begin{align*}
\left\{ a_1c_{ij} + \ldots + a_nc_{nj} &= 0 ; \quad \forall i = 1, m, \forall j = 1, n-m. \\
a_1d_1 + \ldots + a_n d_n &= b_j
\end{align*}
\]

\[
V_j = \begin{pmatrix} c_{ij} \\ c_{nj} \end{pmatrix}, \quad j = 1, n-m, \quad \text{are linearly independent because the solution (S}_2) \text{ is undetermined (n-m)-times.}
\]

\[(?!) \quad V_j, j = 1, n-m, \quad \text{and} \quad d = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}
\]

are linearly independent.

We assume that they are not linearly independent. It follows that

\[
d = s_1 V_1 + \ldots + s_{n-m} V_{n-m} = \begin{pmatrix} s_1 c_{11} + \ldots + s_{n-m} c_{1n-m} \\ \vdots \\ s_1 c_{n1} + \ldots + s_{n-m} c_{nn-m} \end{pmatrix}.
\]

Irrespective of $i = 1, m$:

\[
b_i = a_1d_1 + \ldots + a_n d_n = a_1(s_1 c_{11} + \ldots + s_{n-m} c_{1n-m}) + \ldots + a_n(s_1 c_{n1} + \ldots + s_{n-m} c_{nn-m}) =
\]

\[
= (a_1c_{11} + \ldots + a_n c_{n1})s_1 + \ldots + (a_1c_{1n-m} + \ldots + a_n c_{nn-m})s_{n-m} = 0.
\]

Then, $b_i = 0$, irrespective of $i = 1, m$, contradicts the hypothesis (that there is an $i \in 1, m$, $b_i \neq 0$). It follows that $V_1, \ldots, V_{n-m}, d$ are linearly independent.

\[
\{V_1, \ldots, V_{n-m}\} + d \text{ is a linear variety that contains the solutions of the non-homogeneous system, solutions obtained from (S}_2). \text{ Similarly it follows that}
\]

\[
\{G_1, \ldots, G_r\} + V \text{ is a linear variety containing the solutions of the non-homogeneous system, obtained from (S}_1).
\]

\[
n - m > r \text{ it follows that there are solutions of the system that belong to}
\]

\[
\{V_1, \ldots, V_{n-m}\} + d \text{ and which do not belong to } \{G_1, \ldots, G_r\} + V, \text{ this contradicts the fact that}
\]

\[
(S_1) \text{ is the general solution. Then, it shows that the general solution depends on the } n - m \text{ independent parameters.}
\]

**Theorem 1.** The general solution of a non-homogeneous linear system is equal to the general solution of an associated linear system plus a particular solution of the non-homogeneous system.

**Proof:**

Let’s consider the homogeneous linear solution:
\[
\begin{align*}
\begin{cases}
 a_{11}x_1 + \ldots + a_{1n}x_n = 0 \\
 \vdots \\
 a_{m1}x_1 + \ldots + a_{mn}x_n = 0
\end{cases}, \quad (AX = 0)
\end{align*}
\]

with the general solution:
\[
\begin{align*}
 x_1 &= c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} + d_1 \\
 \vdots \\
 x_n &= c_{n1}k_1 + \ldots + c_{n-m}k_{n-m} + d_n
\end{align*}
\]

and
\[
\begin{align*}
 x_1 &= x_1^0 \\
 \vdots \\
 x_n &= x_n^0
\end{align*}
\]

with the general solution a particular solution of the non-homogeneous linear system \( AX = b \);

\[
\begin{align*}
 x_1 &= c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} + d + x_1^0 \\
 \vdots \\
 x_n &= c_{n1}k_1 + \ldots + c_{n-m}k_{n-m} + d_n + x_n^0
\end{align*}
\]

is a solution of the non-homogeneous linear system.

We note:

\[
A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}
\]

(vector of dimension \( m \)),

\[
K = \begin{pmatrix} k_1 \\ \vdots \\ k_{n-m} \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & \ldots & c_{1n-m} \\ \vdots & \ddots & \vdots \\ c_{n1} & \ldots & c_{n-m} \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}, \quad x^0 = \begin{pmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{pmatrix}
\]

\[
AX = A(Ck + d + x^0) = A(Ck + d) + AX^0 = b + 0 = b
\]

We will prove that irrespective of

\[
\begin{align*}
 x_1 &= y_1^0 \\
 \vdots \\
 x_n &= y_n^0
\end{align*}
\]

there is a particular solution of the non-homogeneous system

\[
\begin{cases}
 k_1 = k_1^0 \in \mathbb{Z} \\
 \vdots \\
 k_{n-m} = k_{n-m}^0 \in \mathbb{Z}
\end{cases}
\]

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We note \( Y^0 = \left( \begin{array}{c} y_1^0 \\ \vdots \\ y_n^0 \end{array} \right) \).

We’ll prove that those \( k^0_j \in \mathbb{Z}, j = 1, n - m \) are those for which \( A\left(CX^0 + d\right) = 0 \) (there are such \( X^0_j \in \mathbb{Z} \) because

\[
\begin{cases}
x_1 = 0 \\
\vdots \\
x_n = 0
\end{cases}
\]

is a particular solution of the homogeneous linear system and \( X = CK + d \) is a general solution of the non-homogeneous linear system

\[
A\left(CK^0 + d + X^0 - Y^0\right) = A\left(CX^0 + d\right) + AX^0 - AY^0 = 0 + b - b = 0 .
\]

**Property 2** The general solution of the homogeneous linear system can be written under the form:

(\(SG\))

\[
\begin{cases}
x_1 = c_1k_1^0 + \ldots + c_{1,m}k_{n-m}^0 \\
\vdots \\
x_n = c_nk_1^0 + \ldots + c_{n,m}k_{n-m}^0
\end{cases}
\]

\(k_j\) is a parameter that belongs to \( \mathbb{Z} \) (with \( d_1 = d_2 = \ldots = d_n = 0 \)).

*Proof:*

(\(SG\)) = general solution. It results that (\(SG\)) is undetermined \( (n - m) \)-times.

Let’s consider that (\(SG\)) is of the form

\[
\begin{cases}
x_1 = c_{11}p_1 + \ldots + c_{1,m}p_{n-m} + d_1 \\
\vdots \\
x_n = c_{n1}p_1 + \ldots + c_{n,m}p_{n-m} + d_n
\end{cases}
\]

with not all \( d_i = 0 \); we’ll prove that it can be written under the form (2); the system has the trivial solution

\[
\begin{cases}
x_1 = 0 \in \mathbb{Z} \\
\vdots \\
x_n = 0 \in \mathbb{Z}
\end{cases}
\]

it results that there are \( p_j \in \mathbb{Z}, j = 1, n - m \),
\[
\begin{align*}
\begin{cases}
  x_1 = c_{11}p_1^0 + \ldots + c_{1n-m}p_{n-m}^0 + d_1 = 0 \\
  \vdots \\
  x_n = c_{n1}p_1^0 + \ldots + c_{nn-m}p_{n-m}^0 + d_n = 0
\end{cases}
\end{align*}
\]

(4)

Substituting \( p_j = k_j + p_j^0, \ j = 1, n - m \) in (3)

\[
\begin{align*}
  k_j \in \mathbb{Z} \quad \Rightarrow \quad p_j \in \mathbb{Z}, \\
  p_j^0 \in \mathbb{Z} \quad \Rightarrow \quad k_j = p_j - p_j^0 \in \mathbb{Z}
\end{align*}
\]

which means that they do not make any restrictions.

It results that

\[
\begin{align*}
\begin{cases}
  x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} + \left( c_{11}p_1^0 + \ldots + c_{1n-m}p_{n-m}^0 + d_1 \right) \\
  \vdots \\
  x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} + \left( c_{n1}p_1^0 + \ldots + c_{nn-m}p_{n-m}^0 + d_n \right)
\end{cases}
\end{align*}
\]

But

\[
\begin{align*}
\begin{cases}
  c_{h1}p_1^0 + \ldots + c_{hn-m}p_{n-m}^0 + d_h = 0, \ h = 1, n \quad \text{(from (4))}.
\end{cases}
\end{align*}
\]

Then the general solution is of the form:

\[
\begin{align*}
\begin{cases}
  x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} \\
  \vdots \\
  x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m}
\end{cases}
\end{align*}
\]

\( k_j \) = parameters \( \in \mathbb{Z}, \ j = 1, n - m \); it results that \( d_1 = d_2 = \ldots = d_n = 0 \).

**Theorem 2.** Let’s consider the homogeneous linear system:

\[
\begin{align*}
\begin{cases}
  a_{11}x_1 + \ldots + a_{1n}x_n = 0 \\
  \vdots \\
  a_{m1}x_1 + \ldots + a_{mn}x_n = 0
\end{cases}
\end{align*}
\]

\( r(A) = m, \ (a_{h1}, \ldots, a_{hm}) = 1, \ h = 1, m \) and the general solution

\[
\begin{align*}
\begin{cases}
  x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} \\
  \vdots \\
  x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m}
\end{cases}
\end{align*}
\]

then

\[
(a_{h1}, \ldots, a_{hi-1}, a_{hi+1}, \ldots, a_{hm}) \left| \left( c_{i1}, \ldots, c_{i,n-m} \right) \right.
\]

irrespective of \( h = 1, m \) and \( i = 1, n \).

**Proof:**

Let’s consider some arbitrary \( h \in 1, m \) and some arbitrary \( i \in 1, n \);

\[
a_{hi}x_1 + \ldots + a_{hi-1}x_{i-1} + a_{hi+1}x_{i+1} + \ldots + a_{hn}x_n = a_{hi}x_i.
\]
Because
\[
\begin{pmatrix}
  a_{h_1}, & \ldots, & a_{h_{i-1}}, & a_{h_{i+1}}, & \ldots, & a_{h_n}
\end{pmatrix}
\begin{pmatrix}
a_{h_i}
\end{pmatrix}
\]

it results that
\[
d = \begin{pmatrix}
  a_{h_1}, & \ldots, & a_{h_{i-1}}, & a_{h_{i+1}}, & \ldots, & a_{h_n}
\end{pmatrix}
\begin{pmatrix}
x_i
\end{pmatrix}
\]

irrespective of the value of \(x_i\) in the vector of particular solutions.

For \(k_2 = k_3 = \ldots = k_{n-m} = 0\) and \(k_1 = 1\) we obtain the particular solution:
\[
\begin{align*}
x_1 &= c_{11} \quad \vdots \quad \vdots \quad x_i &= c_{1i} \; \Rightarrow \; d \mid c_{11} \quad \vdots \quad \vdots \quad x_n &= c_{1n} \\
\end{align*}
\]

For \(k_1 = k_2 = \ldots = k_{n-m-1} = 0\) and \(k_{n-m} = 1\) it results the following particular solution:
\[
\begin{align*}
x_1 &= c_{1n-m} \quad \vdots \quad \vdots \quad x_i &= c_{in-m} \; \Rightarrow \; d \mid c_{in-m} \quad \vdots \quad \vdots \quad x_n &= c_{nn-m} \\
\end{align*}
\]

hence
\[
d \mid c_{ij}, \; j = 1, n-m \Rightarrow d \mid (c_{ij}, \ldots, c_{in-m}).
\]

**Theorem 3.**

If
\[
\begin{align*}
x_1 &= c_{11} k_1 + \ldots + c_{1n-m} k_{n-m} \\
\vdots \\
x_n &= c_{n1} k_1 + \ldots + c_{nn-m} k_{n-m}
\end{align*}
\]

\(k_j = \text{parameters } \in \mathbb{Z}, \; c_{ij} \in \mathbb{Z}\) being given, is the general solution of the homogeneous linear system
\[
\begin{align*}
a_{11} x_1 + \ldots + a_{1n} x_n &= 0 \\
\vdots \\
\end{align*}
\]

\(r(A) = m < n\)

then \((c_{ij}, \ldots, c_{nj}) = 1, \; \forall j = 1, n-m.\)

**Proof:**

We assume, by reduction ad absurdum, that there is \(j_0 \in 1, n-m: (c_{i_{j_0}}, \ldots, c_{n_{j_0}}) = d\)
we consider the maximal co-divisor \(> 0\); we reduce to the case when the maximal co-
divisor is \(-d\) to the case when it is equal to \(d\) (non restrictive hypothesis); then the general solution can be written under the form:

\[
x_1 = c_{11}k_1 + \ldots + c_{nj_0}d_0 + \ldots + c_{1n-m}k_{n-m}
\]

(5)

\[
x_n = c_{n1}k_1 + \ldots + c_{nj_0}d_0 + \ldots + c_{an-m}k_{n-m}
\]

where \(d = (c_{ij}, \ldots, c_{nj_0})\), \(c_{ij_0} = d \cdot c_{ij_0}'\) and \((c_{ij_0}', \ldots, c_{nj_0}') = 1\).

We prove that

\[
x_1 = c_{ij_0}
\]

\[
\vdots
\]

\[
x_n = c_{nj_0}
\]

is a particular solution of the homogeneous linear system.

We’ll note:

\[
C = \begin{pmatrix}
c_{11} & \ldots & c_{ij_0} & d & \ldots & c_{1n-m}

\vdots & \vdots & \vdots & \vdots & \vdots & \vdots

c_{n1} & \ldots & c_{nj_0} & d & \ldots & c_{an-m}
\end{pmatrix},
\]

\[
k = \begin{pmatrix}
k_1

\vdots

k_{j_0}

\vdots

k_{n-m}
\end{pmatrix}
\]

\[
x = C \cdot k\]

the general solution.

We know that \(AX = 0 \Rightarrow A(CK) = 0\), \(A = \begin{pmatrix}
a_{11} & \ldots & a_{1n}

\vdots

a_{n1} & \ldots & a_{nn}
\end{pmatrix}\).

We assume that the principal variables are \(x_1, \ldots, x_m\) (if not, we have to renumber). It follows that \(x_{m+1}, \ldots, x_n\) are the secondary variables.

For \(k_1 = \ldots = k_{j_0-1} = k_{j_0+1} = \ldots = k_{n-m} = 0\) and \(k_{j_0} = 1\) we obtain a particular solution of the system

\[
\begin{pmatrix}
x_1 = c_{11}d

\vdots

x_n = c_{nj_0}d
\end{pmatrix} \Rightarrow 0 = A \begin{pmatrix}
c_{1j_0}d

\vdots
nc_{nj_0}d
\end{pmatrix} = d \cdot A \begin{pmatrix}
c_{1j_0}

\vdots
nc_{nj_0}
\end{pmatrix} \Rightarrow A \begin{pmatrix}
x_1 = c_{1j_0}

\vdots
x_n = c_{nj_0}
\end{pmatrix} = 0 \Rightarrow :
\]

is the particular solution of the system.

We’ll prove that this particular solution cannot be obtained by

\[
x_1 = c_{11}k_1 + \ldots + c_{1j_0}d_0 + \ldots + c_{1n-m}k_{n-m} = c_{1j_0}
\]

(6)

\[
x_n = c_{n1}k_1 + \ldots + c_{nj_0}d_0 + \ldots + c_{n-m}k_{n-m} = c_{nj_0}
\]
\[
\begin{bmatrix}
\vdots \\
\vdots \\
\end{bmatrix}
\begin{align*}
\begin{bmatrix}
\vdots \\
\vdots \\
\end{bmatrix} & = \begin{bmatrix} c_{m+1,1} & \cdots & c_{m+1,j} & \cdots & c_{m+1,n-m} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix} \begin{bmatrix}
\vdots \\
\vdots \\
\end{bmatrix} \\
\begin{bmatrix}
\vdots \\
\vdots \\
\end{bmatrix} & = \begin{bmatrix} c_{h,1} & \cdots & c_{nj} & \cdots & c_{n,n-m} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix} \begin{bmatrix}
\vdots \\
\vdots \\
\end{bmatrix}
\end{align*}
\]

\[
\Rightarrow k_{j_0} = \frac{1}{d} \not\in \mathbb{Z}
\]

(because \(d \neq 1\)).

It is important to point out the fact that those \(k_j = k_j^0, \ j = 1, n - m\), that satisfy the system (7) also satisfy the system (6), because, otherwise (6) would not satisfy the definition of the solution of a linear system of equations (i.e., considering the system (7) the hypothesis was not restrictive). From \(X_{j_0} \in \mathbb{Z}\) follows that (6) is not the general solution of the homogeneous linear system contrary to the hypothesis; then \((c_{j_1}, \ldots, c_{nj}) = 1\), irrespective of \(j = 1, n - m\).

**Property 3.** Let’s consider the linear system

\[
\begin{align*}
\begin{bmatrix}
\vdots \\
\vdots \\
\end{bmatrix} & = \begin{bmatrix}
\vdots \\
\vdots \\
\end{bmatrix} \\
\begin{bmatrix}
\vdots \\
\vdots \\
\end{bmatrix} & = \begin{bmatrix}
\vdots \\
\vdots \\
\end{bmatrix}
\end{align*}
\]

\(a_{ij}, b_i \in \mathbb{Z}, \ r(A) = m < n, \ x_j = \text{unknowns} \in \mathbb{Z}\)

Resolved in \(\mathbb{R}\), we obtain

\[
\begin{align*}
\begin{bmatrix}
\vdots \\
\vdots \\
\end{bmatrix} & = \begin{bmatrix}
\vdots \\
\vdots \\
\end{bmatrix} \\
\begin{bmatrix}
\vdots \\
\vdots \\
\end{bmatrix} & = \begin{bmatrix}
\vdots \\
\vdots \\
\end{bmatrix}
\end{align*}
\]

where \(f_i\) are linear functions of the form:

\[
f_i = \frac{c'_{m+1} x_{m+1} + \cdots + c'_n x_n + e_i}{d_i},
\]

where \(c'_{m+j}, d_i, e_i \in \mathbb{Z}; \ i = 1, m, \ j = 1, n - m\).

If \(\frac{e_i}{d_i} \in \mathbb{Z}\) irrespective of \(i = 1, m\) then the linear system has integer solution.

**Proof:**

For \(1 \leq i \leq m, \ x_i \in \mathbb{Z}\), then \(f_j \in \mathbb{Z}\). Let’s consider
\[
\begin{align*}
\begin{cases}
x_{m+1} &= u_{m+1}k_{m+1} \\
\vdots \\
x_n &= u_nk_n \\
\vdots \\
x_1 &= v_1^{m+1}k_{m+1} + \ldots + v_n^{m}k_n + \frac{e_i}{d_i} \\
\vdots \\
x_m &= v_n^{m}k_{m+1} + \ldots + v_n^{m}k_n + \frac{e_m}{d_m}
\end{cases}
\end{align*}
\]

a solution, where \( u_{m+1} \) is the maximal co-divisor of the denominators of the fractions \( \frac{c_{m+j}^i}{d_i}, \ i = 1, m, \ j = 1, n - m \) calculated after their complete simplification.

\[
v_{m+j}^i = \frac{c_{m+j}^i u_{m+j}}{d_i} \in \mathbb{Z} \quad \text{is a } (n - m)\text{-times undetermined solution which depends on } n - m \text{ independent parameters } (k_{m+1}, \ldots, k_n) \text{ but is not a general solution.}
\]

**Property 4.** Under the conditions of property 3, if there is an

\[
i_0 \in \{1, m\} : f_i = u_{m+i}x_{m+1} + \ldots + u_nx + \frac{e_i}{d_i} \quad \text{with } u_{m+i} \in \mathbb{Z}, \ j = 1, n - m, \text{ and } \frac{e_i}{d_i} \notin \mathbb{Z} \text{ then the system does not have integer solution.}
\]

**Proof:**

\[\forall x_{m+1}, \ldots, x_n \in \mathbb{Z}, \text{ it results that } x_{i_0} \notin \mathbb{Z}.\]

**Theorem 4.** Let’s consider the linear system

\[
\begin{align*}
\begin{cases}
a_{i_1}x_1 + \ldots + a_{i_m}x_n &= b_1 \\
\vdots \\
a_{m_1}x_1 + \ldots + a_{m_n}x_n &= b_m
\end{cases}
\end{align*}
\]

\(a_j, b_i \in \mathbb{Z}, \ x_j = \text{unknowns } \in \mathbb{Z}, \ r(A) = m < n.\) If there are indices \( 1 \leq i_1 < \ldots < i_m \leq n, \)

\(i_h \in \{1, 2, \ldots, n\}, \ h = 1, m,\) with the property:

\[
\Delta = \begin{vmatrix}
a_{i_{i_1}} & \ldots & a_{i_{i_m}} \\
\vdots & \ddots & \vdots \\
a_{m_{i_{i_1}}} & \ldots & a_{m_{i_m}}
\end{vmatrix} \neq 0 \text{ and}
\]

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\[ \Delta x_i = \begin{vmatrix} b_i & a_{i1} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n1} & \cdots & a_{nn} \end{vmatrix} \] is divided by \( \Delta \)

\[ \Delta x_{ih} = \begin{vmatrix} a_{i1} & \cdots & a_{ih-1} & b_i \\ \vdots & \ddots & \vdots & \vdots \\ a_{mi1} & \cdots & a_{mih-1} & b_m \end{vmatrix} \] is divided by \( \Delta \)

then the system has integer number solutions.

**Proof:**

We use property 3

\[ d_i = \Delta, \ i = 1, m; \ e_{ih} = \Delta x_{ih}, \ h = 1, m \]

**Note 1.** It is not true in the reverse case.

**Consequence 1.** Any homogeneous linear system has integer number solutions (besides the trivial one); \( r(A) = m < n \).

**Proof:**

\[ \Delta x_{ih} = 0 : \Delta, \ \text{irrespective of} \ h = 1, m \]

**Consequence 2.** If \( \Delta = \pm 1 \), it follows that the linear system has integer number solutions.

**Proof:**

\[ \Delta x_{ih} : (\pm 1), \ \text{irrespective of} \ h = 1, m \]

\[ \Delta x_{ih} \in \mathbb{Z} \]