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K-Divisibility and K-Strong
Divisibility Sequences

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K-Divisibility and K-Strong Divisibility Sequences

A sequence of rational integers \( g \) is called a divisibility sequence if and only if
\[
n | m \Rightarrow g(n) | g(m)
\]
for all positive integers \( n, m \). [See [3] and [4]]

Also, \( g \) is called a strong divisibility sequence if and only if
\[
(g(n), g(m)) = g((n, m))
\]
for all positive integers \( n, m \). [See [1], [2], [3], [4] and [5]]

Of course, it is easy to show that the results of the Smarandache function \( S(n) \) is neither a divisibility or a strong divisibility sequence because \( 4|20 \) but \( S(4) = 4 \) does not divide \( 5 = S(20) \), and \( (S(4), S(20)) = (4, 5) = 1 \neq 4 = S(4) = S((4, 20)) \).

a) However, is there an infinite subsequence of integers \( M = \{m_1, m_2, \ldots \} \) such that \( S \) is a divisibility sequence on \( M \)?

b) If \( P(p_1, p_2, \ldots) \) is the set of prime numbers, the \( S \) is not a strong divisibility sequence on \( P \), because for \( i \neq j \) we have
\[
(S(p_i), S(p_j)) = (p_i, p_j) = 1 \neq 0 = S(1) = S((p_i, p_j)).
\]

And the same question can be asked about \( P \) as was asked in part (a).

We introduce the following two notions, which are generalizations of a "divisibility sequence" and "strong divisibility sequence" respectively.

1) A \( k \)-divisibility sequence, where \( k \geq 1 \) is an integer, is defined in the following way:
If \( n | m \Rightarrow g(n) | g(m) \Rightarrow g(g(n)) | g(g(m)) \Rightarrow \ldots \Rightarrow g(\ldots (g(n)) \ldots) | g(\ldots (g(m)) \ldots) \) for all \( k \) times \( k \) times

\( n, m \) for all positive integers \( n, m \).

For example, \( g(n) = n! \) is a \( k \)-divisibility sequence.

Also: any constant sequence is a \( k \)-divisibility sequence.

2) A \( k \)-strong divisibility sequence, where \( k \geq 1 \) is an integer, is defined in the following way:
If \( g(n_1), g(n_2), \ldots, g(n_k)) = g((n_1, n_2, \ldots, n_k)) \) for all positive integers \( n_1, n_2, \ldots, n_k \).

For example, \( g(n) = 2n \) is a \( k \)-strong divisibility sequence, because \( (2n_1, 2n_2, \ldots, 2n_k) = 2 \cdot (n_1, n_2, \ldots, n_k) = g((n_1, n_2, \ldots, n_k)) \).
Remarks: If $g$ is a divisibility sequence and we apply its definition $k$-times, we get that $g$ is a $k$-divisibility sequence for any $k \geq 1$. The converse is also true. If $g$ is a $k$-strong divisibility sequence, $k \geq 2$, then $g$ is a strong divisibility sequence. This can be seen by taking the definition of a $k$-strong divisibility sequence and replacing $n$ by $n_1$ and all $n_2, \ldots, n_k$ by $m$ to obtain $(g(n), g(m), \ldots, g(m)) = g(n, m, \ldots, m)$ or $(g(n), g(m)) = g(n, m, \ldots, m)$.

The converse is also true, as

$$(n, n_2, \ldots, n_k) = ((n, n_2), n_3), \ldots, n_k).$$

Therefore, we found that:

a) The divisibility sequence notion is equivalent to a $k$-divisibility sequence, or a generalization of a notion is equivalent to itself.

Is there any paradox or dilemma?

b) The strong divisibility sequence is equivalent to the $k$-strong divisibility sequence notion.

As before, a generalization of a notion is equivalent to itself.

Again, is there any paradox or dilemma?

References


