Lemoine Circles

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In this article, we get to Lemoine's circles in a different manner than the known one.

Theorem 1.

Let $ABC$ a triangle and $K$ its simedian center. We take through $K$ the parallel $A_1A_2$ to $BC$, $A_1 \in (AB)$, $A_2 \in (AC)$; through $A_2$ we take the antiparallels $A_2B_1$ to $AB$ in relation to $CA$ and $CB$, $B_1 \in (BC)$; through $B_1$ we take the parallel $B_1B_2$ to $AC$, $B_2 \in AB$; through $B_2$ we take the antiparallels $B_1C_1$ to $BC$, $C_1 \in (AC)$, and through $C_1$ we take the parallel $C_1C_2$ to $AB$, $C_1 \in (BC)$. Then:

i. $C_2A_1$ is an antiparallel of $AC$;

ii. $B_1B_2 \cap C_1C_2 = \{K\}$;

iii. The points $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclic (the first Lemoine circle).

Proof.

i. The quadrilateral $BC_2KA$ is a parallelogram, and its center, i.e. the middle of the segment $(C_2A_1)$, belongs to the simedian $BK$; it follows that $C_2A_2$ is an antiparallel to $AC$ (see Figure 1).

ii. Let $\{K'\} = A_1A_2 \cap B_1B_2$, because the quadrilateral $K'B_1CA_2$ is a parallelogram; it follows that $CK'$ is a simedian; on the other hand, $CK$ is a simedian, and since $K, K' \in A_1A_2$, it follows that we have $K' = K$. 

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iii. $B_2C_1$ being an antiparallel to $BC$ and $A_1A_2 \parallel BC$, it means that $B_2C_1$ is an antiparallel to $A_1A_2$, so the points $B_2, C_1, A_2, A_1$ are concyclic.

From $B_1B_2 \parallel AC, \measuredangle B_2C_1A \equiv \measuredangle ABC, \measuredangle B_1A_2C \equiv \measuredangle ABC$ we get that the quadrilateral $B_2C_1A_2B_1$ is an isosceles trapezoid, so the points $B_2, C_1, A_2, B_1$ are concyclic.

Analogously, it can be shown that the quadrilateral $C_2B_1A_2A_1$ is an isosceles trapezoid, therefore the points $C_2, B_1, A_2, A_1$ are concyclic.

From the previous three quartets of concyclical points, it results the concyclicity of the points belonging to the first Lemoine circle.

Theorem 2.
In the scalene triangle $ABC$, let $K$ be the simedian center. We take from $K$ the antiparallel $A_1A_2$ to $BC; A_1 \in AB, A_2 \in AC$; through $A_2$ we build $A_2B_1 \parallel AB$; $B_1 \in (BC)$, then through $B_1$ we build $B_1B_2$ the antiparallel to $AC, B_2 \in (AB)$, and through $B_2$ we build $B_2C_1 \parallel BC, C_1 \in AC$, and, finally, through $C_1$ we take the antiparallel $C_1C_2$ to $AB, C_2 \in (BC)$. Then:
i. \( C_2 A_1 \parallel AC \);

ii. \( B_1 B_2 \cap C_1 C_2 = \{ K \} \);

iii. The points \( A_1, A_2, B_1, B_2, C_1, C_2 \) are concyclical (the second Lemoine circle).

**Proof.**

i. Let \( \{ K' \} = A_1 A_2 \cap B_1 B_2 \), having \( \angle AA_1 A_2 = \angle ACB \) and \( \angle BB_1 B_2 \equiv \angle BAC \) because \( A_1 A_2 \) and \( B_1 B_2 \) are antiparallels to \( BC, AC \), respectively, it follows that \( \angle K' A_1 B_2 \equiv \angle K' B_2 A_1 \); so \( K'A_1 = K'B_2 \); having \( A_1 B_2 \parallel B_1 A_2 \) as well, it follows that also \( K'A_2 = K'B_1 \), so \( A_1 A_2 = B_1 B_2 \). Because \( C_1 C_2 \) and \( B_1 B_2 \) are antiparallels to \( AB \) and \( AC \), we have \( K''C_2 = K''B_1 \); we noted \( \{ K'' \} = B_1 B_2 \cap C_1 C_2 \); since \( C_1 B_2 \parallel B_1 C_2 \), we have that the triangle \( K''C_1 B_2 \) is also isosceles, therefore \( K''C_1 = C_1 B_2 \), and we get that \( B_1 B_2 = C_1 C_2 \). Let \( \{ K''' \} = A_1 A_2 \cap C_1 C_2 \); since \( A_1 A_2 \) and \( C_1 C_2 \) are antiparallels to \( BC \) and \( AB \), we get that the triangle \( K'''A_2 C_1 \) is isosceles, so \( K'''A_2 = K'''C_1 \), but \( A_1 A_2 = C_1 C_2 \) implies that \( K'''C_2 = K'''A_1 \), then \( \angle K'''A_1 C_2 \equiv \angle K'''A_2 C_1 \) and, accordingly, \( C_2 A_1 \parallel AC \).

![Figure 2](image)

ii. We noted \( \{ K' \} = A_1 A_2 \cap B_1 B_2 \); let \( \{ X \} = B_2 C_1 \cap B_1 A_2 \); obviously, \( BB_1 X B_2 \) is a parallelogram; if \( K_0 \) is the middle of \( (B_1 B_2) \), then \( BK_0 \) is a simedian,
since $B_1B_2$ is an antiparallel to $AC$, and the middle of the antiparallels of $AC$ are situated on the simedian $BK$. If $K_0 \neq K$, then $K_0K \parallel A_1B_2$ (because $A_1A_2 = B_1B_2$ and $B_1A_2 \parallel A_1B_2$), on the other hand, $B, K_0, K$ are collinear (they belong to the simedian $BK$), therefore $K_0K$ intersects $AB$ in $B$, which is absurd, so $K_0 = K$, and, accordingly, $B_1B_2 \cap A_1A_2 = \{K\}$. Analogously, we prove that $C_1C_2 \cap A_1A_2 = \{K\}$, so $B_1B_2 \cap C_1C_2 = \{K\}$.

iii. $K$ is the middle of the congruent antiparallels $A_1A_2, B_1B_2, C_1C_2$, so $KA_1 = KA_2 = KB_1 = KB_2 = KC_1 = KC_2$. The simedian center $K$ is the center of the second Lemoine circle.

Remark.

The center of the first Lemoine circle is the middle of the segment $[OK]$, where $O$ is the center of the circle circumscribed to the triangle $ABC$. Indeed, the perpendiculars taken from $A, B, C$ on the antiparallels $B_2C_1, A_1C_2, B_1A_2$ respectively pass through $O$, the center of the circumscribed circle (the antiparallels have the directions of the tangents taken to the circumscribed circle in $A, B, C$). The mediatrix of the segment $B_2C_1$ pass though the middle of $B_2C_1$, which coincides with the middle of $AK$, so is the middle line in the triangle $AKO$ passing through the middle of $(OK)$. Analogously, it follows that the mediatrix of $A_1C_2$ pass through the middle $L_1$ of $[OK]$.

Bibliography.

