THE POLAR OF A POINT With Respect TO A CIRCLE

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In this article we establish a connection between the notion of the symmedian of a triangle and the notion of polar of a point in rapport to a circle.

We’ll prove for beginning two properties of the symmedians.

**Lemma 1**

If in triangle ABC inscribed in a circle, the tangents to this circle in the points B and C intersect in a point S, then AS is symmedian in the triangle ABC.

**Proof**

We’ll note L the intersection point of the line AS with BC (see fig. 1).

![Fig. 1](image-url)

We have
\[
\frac{\text{Aria } \triangle ABL}{\text{Aria } \triangle ACL} = \frac{BL}{LC} = \frac{\text{Aria } \triangle BSL}{\text{Aria } \triangle CSL}
\]

It result
\[
\frac{\text{Aria } \triangle ABS}{\text{Aria } \triangle ACS} = \frac{BL}{LC}
\]

We observe that
\[
m(\angle ABS) = m(\hat{B}) + m(\hat{A}) \quad \text{and} \quad m(\angle ACS) = m(\hat{C}) + m(\hat{A})
\]

We obtain that
\[
\sin(m(\angle ABS)) = \sin C \quad \text{and} \quad \sin(m(\angle ACS)) = \sin B
\]

We have also
\[
\frac{\text{Aria } \triangle ABS}{\text{Aria } \triangle ACS} = \frac{AB \cdot SB \cdot \sin C}{AC \cdot SC \cdot \sin B} = \frac{BL}{LC}
\]

From the sinus’ theorem it results

\[\text{(1)}\]
\[\text{(2)}\]
\[
\frac{\sin C}{\sin B} = \frac{AB}{AC}
\]  
(3)

The relations (2) and lead us to the relation
\[
\frac{BL}{LC} = \left(\frac{AB}{AC}\right)^2,
\]
which shows that \(AS\) is symmedian in the triangle \(ABC\).

**Observations**

1. The proof is similar if the triangle \(ABC\) is obtuse.
2. If \(ABC\) is right triangle in \(A\), the tangents in \(B\) and \(C\) are parallel, and the symmedian from \(A\) is the height from \(A\), and, therefore, it is also parallel with the tangents constructed in \(B\) and \(C\) to the circumscribed circle.

**Definition 1**

The points \(A, B, C, D\) placed, in this order, on a line \(d\) form a harmonic division if and only if
\[
\frac{AB}{AD} = \frac{CB}{CD}
\]

**Lemma 2**

If in the triangle \(ABC\), \(AL\) is the interior symmedian \(L\in BC\), and \(AP\) is the external median \(P\in BC\), then the points \(P, B, L, C\) form a harmonic division.

**Proof**

It is known that the external symmedian \(AP\) in the triangle \(ABC\) is tangent in \(A\) to the circumscribed circle (see fig. 2), also, it can be proved that:
\[
\frac{PB}{PC} = \left(\frac{AB}{AC}\right)^2
\]
(1)

but
\[
\frac{LB}{LC} = \left(\frac{AB}{AC}\right)^2
\]
(2)

![Fig. 2](image)

From the relations (1) and (2) it results
\[
\frac{PB}{PC} = \frac{LB}{LC},
\]
Which shows that the points \(P, B, L, C\) form a harmonic division.
Definition 2
If $P$ is a point exterior to circle $C(0,r)$ and $B, C$ are the intersection points of the circle with a secant constructed through the point $P$, we will say about the point $Q \in (BC)$ that it is the harmonic conjugate of the point $P$ in rapport to the circle $C(0,r)$.

Observation
In the same conjunction, the point $P$ is also the conjugate of the point $Q$ in rapport to the circle (see fig. 3).

Definition 3
The set of the harmonic conjugates of a point in rapport with a given circle is called the polar of that point in rapport to the circle.

Theorem
The polar of an exterior point to the circle is the circle’s cord determined by the points of tangency with the circle of the tangents constructed from that point to the circle.

Proof
Let $P$ an exterior point of the circle $C(0,r)$ and $M, N$ the intersections of the line $PO$ with the circle (see fig. 4).

We note $T$ and $V$ the tangent points with the circle of the tangents constructed from the point $P$ and let $Q$ be the intersection between $MN$ and $TV$.

Obviously, the triangle $MTN$ is a right triangle in $T$, $TQ$ is its height (therefore the interior symmedian, and $TP$ is the exterior symmedian, and therefore the points $P, M, Q, N$ form a harmonic division, (Lemma 2)). Consequently, $Q$ is the harmonic conjugate of $P$ in rapport to the circle and it belongs to the polar of $P$ in rapport to the circle.

We’ll prove that $(TV)$ is the polar of $P$ in rapport with the circle. Let $M'N'$ be the intersections of a random secant constructed through the point $P$ with the circle, and $X$ the intersection of the tangents constructed in $M'$ and $N'$ to the circle.

In conformity to Lemma 1, the line $XT$ is for the triangle $M'TN'$ the interior symmedian, also $TP$ is for the same triangle the exterior symmedian.

If we note $Q'$ the intersection point between $XT$ and $M'N'$ it results that the point $Q'$ is the harmonic conjugate of the point $P$ in rapport with the circle, and consequently, the point $Q'$ belongs to the polar $P$ in rapport to the circle.
For the triangle VM'N', according to Lemma 1, the line VX is the interior symmedian and VP is for the same triangle the external symmedian. It will result, according to Lemma 2, that if \( \{Q', Q''\} = VX \cap M'N' \), the point Q'' is the harmonic conjugate of the point P in rapport to the circle. Because the harmonic conjugate of a point in rapport with a circle is a unique point, it results that Q'=Q''. Therefore the points V, T, X are collinear and the point Q' belongs to the segment (TV).

**Reciprocal**

If \( Q_1 \in (TV) \) and \( PQ_1 \) intersect the circle in \( M_1 \) and \( N_1 \), we much prove that the point \( Q_1 \) is the harmonic conjugate of the point P in rapport to the circle.

Let \( X_1 \) the intersection point of the tangents constructed from \( M_1 \) and \( N_1 \) to the circle. In the triangle \( M_1TN_1 \) the line \( X_1T \) is interior symmedian, and the line \( TP \) is exterior symmedian. If \( \{Q'_1, Q''_1\} = X_1T \cap M_1N_1 \), then \( P, M_1, Q'_1, N_1 \) form a harmonic division.

Similarly, in the triangle \( M_1VN_1 \) the line \( VX_1 \) is interior symmedian, and \( VP \) exterior symmedian. If we note \( \{Q''_1, Q'_1\} = VX_1 \cap M_1N_1 \), it results that the point \( Q'_1 \) is the harmonic conjugate of the point P in rapport to \( M_1 \) and \( N_1 \). Therefore, we obtain \( Q'_1 = Q''_1 \). On the other side, \( X_1, T, Q'_1 \) and \( V, X_1, Q''_1 \) are collinear, but \( Q'_1 = Q''_1 \), it result that \( X_1, T, Q'_1 \), \( V \) are collinear, and then \( Q'_1 = Q_1 \), therefore \( Q_1 \) is the conjugate of \( P \) in rapport with the circle.