From a problem of geometrical construction to the Carnot circles

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In this article we’ll give solution to a problem of geometrical construction and we’ll show the connection between this problem and the theorem relative to Carnot’s circles.

Let \( ABC \) a given random triangle. Using only a compass and a measuring line, construct a point \( M \) in the interior of this triangle such that the circumscribed circles to the triangles \( MAB \) and \( MAC \) are congruent.

**Construction**

We’ll start by assuming, as in many situations when we have geometrical constructions, that the construction problem is resolved.

Let \( M \) a point in the interior of the triangle \( ABC \) such that the circumscribed circles to the triangles \( MAB \) and \( MAC \) are congruent.

We’ll note \( O_c \) and \( O_b \) the centers of these triangles, these are the intersections between the mediator of the segments \( [AB] \) and \( [AC] \). The quadrilateral \( AO_cMO_b \) is a rhomb (therefore \( M \) is the symmetrical of the point \( A \) in rapport to \( O_bO_c \) (see Fig. 1).

**A. Step by step construction**

We’ll construct the mediators of the segments \([AB]\) and \([AC]\), let \( R, S \) be their intersection points with \([AB]\) respectively \([AC]\). (We suppose that \( AB < AC \), therefore \( AR < AS \).) With the compass in \( A \) and with the radius larger than \( AS \) we construct a circle which intersects \( OR \) in \( O_c \) and \( O_c \) respectively \( OS \) in \( O_b \) and \( O_b \). - \( O \) being the circumscribed circle to the triangle \( ABC \).

Now we construct the symmetric of the point \( A \) in rapport to \( O_cO_b \); this will be the point \( M' \), and if we construct the symmetric of the point \( A \) in rapport to \( O_cO_b \) we obtain the point \( M' \).

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*Lazare Carnot (1753 – 1823)*, French mathematician, mechanical engineer and political personality (Paris).
B. Proof of the construction

Because $AO_c = AO_b$ and $M$ is the symmetric of the point $A$ in rapport of $O_cO_b$, it results that the quadrilateral $AO_cMO_b$ will be a rhombus, therefore $O_cA = O_cM$ and $O_bA = O_bM$. On the other hand, $O_c$ and $O_b$ being perpendicular points of $AB$ respectively $AC$, we have $O_cA = O_cB$ and $O_bA = O_bC$, consequently 

$$O_cA = O_cM = O_bA = O_bM = O_bC,$$

which shows that the circumscribed circles to the triangles $MAB$ and $MAC$ are congruent.

Similarly, it results that the circumscribed circles to the triangles $ABM'$ and $ACM'$ are congruent, more so, all the circumscribed circles to the triangles $MAB, MAC, M'AB, M'AC$ are congruent.

As it can be in the Fig. 2, the point $M'$ is in the exterior of the triangle $ABC$.

Discussion

We can obtain, using the method of construction shown above, an infinity of pairs of points $M$ and $M'$, such that the circumscribed circles to the triangles $MAB, MAC, M'AB, M'AC$ will be congruent. It seems that the point $M'$ is in the exterior of the triangle $ABC$.
Observation

The points $M$ from the exterior of the triangle $ABC$ with the property described in the hypothesis are those that belong to the arch $BC$, which does not contain the vertex $A$ from the circumscribed circle of the triangle $ABC$.

Now, we’ll try to answer to the following:

Questions

1. Can the circumscribed circles to the triangles $MAB, MAC$ with $M$ in the interior of the triangle $ABC$ be congruent with the circumscribed circle of the triangle $ABC$?
2. If yes, then, what can we say about the point $M$?

Answers

1. The answer is positive. In this hypothesis we have $OA = AO_b = AO_c$ and it results also that $O_c$ and $O_b$ are the symmetrical of $O$ in rapport to $AB$ respectively $AC$. The point $M$ will be, as we showed, the symmetric of the point $A$ in rapport to $O_cO_b$.

   The point $M$ will be also the orthocenter of the triangle $ABC$. Indeed, we prove that the symmetric of the point $A$ in rapport to $O_cO_b$ is $H$ which is the orthocenter of the triangle $ABC$.

   Let $RS$ the middle line of the triangle $ABC$. We observe that $RS$ is also middle line in the triangle $OO_bO_c$, therefore $O_bO_c$ is parallel and congruent with $BC$, therefore it results that $M$ belongs to the height constructed from $A$ in the triangle $ABC$. We’ll note $T$ the middle of $[BC]$, and let $R$ the radius of the circumscribed circle to the triangle $ABC$; we have

   \[ OT = \sqrt{R^2 - \frac{a^2}{4}}, \text{ where } a = BC. \]

   If $P$ is the middle of thesegment $[AM]$, we have

   \[ AP = \sqrt{R^2 - PO_b^2} = \sqrt{R^2 - \frac{a^2}{4}}. \]

   From the relation $AM = 2 \cdot OT$ it results that $M$ is the orthocenter of the triangle $ABC$, ($AH = 2OT$).

   The answers to the questions 1 and 2 can be grouped in the following form:

Proposition

There is only one point in the interior of the triangle $ABC$ such that the circumscribed circles to the triangles $MAB, MAC$ and $ABC$ are congruent. This point is the orthocenter of the triangle $ABC$.

Remark
From this proposition it practically results that the unique point $M$ from the interior of the right triangle $ABC$ with the property that the circumscribed circles to the triangles $MAB, MAC, MBC$ are congruent with the circumscribed circle to the triangle is the point $H$, the triangle’s orthocenter.

**Definition**
If in the triangle $ABC$, $H$ is the orthocenter, then the circumscribed circles to the triangles $HAB, HAC, HBC$ are called Carnot circles.

We can prove, without difficulty the following:

**Theorem**
The Carnot circles of a triangle are congruent with the circumscribed circle to the triangle.

**References**