From Newton’s Theorem to a Theorem of the Inscribable Octagon

Prof. Ion Pătraşcu, The Fraţii Buzeşti National College, Craiova, Romania
Prof. Florentin Smarandache, The University of New Mexico, U.S.A.

In this article we’ll prove the Newton’s theorem relative to the circumscribed quadrilateral, we’ll transform it through duality, and we obtain another theorem which is true for an inscribable quadrilateral, which transformed through duality, we’ll obtain a theorem which is true for a circumscribable octagon.

Theorem 1 (I. Newton)
In a circumscribable quadrilateral its diagonals and the cords determined by the contact points of the opposite sides of the quadrilateral with the circumscribed circle are four concurrent lines.

Proof

Fig. 1
We constructed the circles $O_1, O_2, O_3, O_4$ tangent to the extensions of the quadrilateral $ABCD$ such that

$$A_4M = A_4N = B_1P = B_1Q = C_1R = C_1S = D_1U = D_1V$$

See Fig. 1.

From $A_4M = A_4N = C_1R = C_1S$ it results that the points $A_4$ and $C_1$ have equal powers in relation to the circles $O_1$ and $O_3$, therefore $A_4C_1$ is the radical axis of these circles. Similarly $B_1D_1$ is the radical axis of the circles $O_2$ and $O_4$.

Let $I \in A_4C_1 \cap B_1D_1$. The point $I$ has equal powers in rapport to circles $O_1, O_2, O_3, O_4$. Because $BA_4 = BB_1$ from $B_1P = A_4N$ it results that $BP = BN$, similarly, from $DD_1 = DC_1$ and $D_1V = C_1S$ it results that $DV = DS$, therefore $B$ and $D$ have equal powers in rapport with the circles $O_3$ and $O_4$, which shows that $BD$ is the radical axis of these circles. Consequently, $I \in BD$, similarly it results that $I \in AC$, and the proof is complete.

**Theorem 2.**

In an inscribed quadrilateral in which the opposite sides intersect, the intersection points of the tangents constructed to the circumscribed circle with the opposite vertexes and the points of intersection of the opposite sides are collinear.

**Proof**

We’ll prove this theorem applying the configuration from the Newton theorem, o transformation through duality in rapport with the circle inscribed in the quadrilateral. Through this transformation to the lines $AB, BC, CD, DA$ will correspond, respectively, the points $A_4, B_1, C_1, D_1$ their pols. Also to the lines $A_4B_1, B_1C_1, C_1D_1, D_1A_4$ correspond, respectively, the points $B, C, D, A$. We note $X \in AB \cap CD$ and $Y \in AD \cap BC$, these points correspond, through the considered duality, to the lines $A_4C_1$ respectively $B_1D_1$. If $I \in A_4C_1 \cap B_1D_1$ then to the point $I$ corresponds line $XY$, its polar.

To line $BD$ corresponds the point $Z \in A_4D_1 \cap C_1B_1$.

To line $AC$ corresponds the point $T \in A_4D_1 \cap C_1B_1$.

To point $\{I\} = BD \cap AC$ corresponds its polar $ZT$.

We noticed that to the point $I$ corresponds the line $XY$, consequently the points $X, Y, Z, T$ are collinear.

We obtained that the quadrilateral $A_4B_1C_1D_1$, inscribed in a circle has the property that if $A_4D_1 \cap C_1B_1 = \{Z\}$, $A_4D_1 \cap C_1B_1 = \{T\}$, the tangent in $A_4$ and the tangent in $C_1$ intersect in the point $X$; the tangent in $B_1$ and the tangent in $D_1$ intersect in $Y$, then $X, Y, Z, T$ are collinear (see Fig. 2).

**Theorem 3.**

In a circumscribed octagon, the four cords, determined by the octagon’s contact points with the circle of the octagon opposite sides, are concurrent.
Proof

We’ll transform through reciprocal polar the configuration in figure 3.

To point $Z$ corresponds through this transformation the line determined by the tangency points with the circle of the tangents constructed from $Z$ - its polar; to the point $Y$ it corresponds the line determined by the contact points of the tangents constructed from $T$ at the circle; to the point $X$ corresponds its polar $A_iC_i$.

To point $A_i$ corresponds through duality the tangent $A_iX$, also to the points $B_i, C_i, D_i$ correspond the tangents $B_iY, C_iT, D_iZ$.

Fig. 2

These four tangents together with the tangents constructed from $X$ and $Y$ (also four) will contain the sides of an octagon circumscribed to the given circle.
In this octagon $A_1C_1$ and $B_1D_1$ will connect the contact points of two pairs of sides opposed to the circle, the other two cords determined by the contact points of the opposite sides of the octagon with the circle will be the polar of the points $Z$ and $T$.

Because the transformation through reciprocal polar will make that to collinear points will correspond concurrent lines, these lines are the cords from our initial statement.

**Observation**
In figure 3 we represented an octagon $ABCDEFGH$ circumscribed to a circle.
As it can be observed the cords $MR, NS, PT, QU$ are concurrent in a point notated $W$

**References**
